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LINEAR PROJECTIONS WHICH IMPLEMENT BALAYAGE IN FOURIER TRANSFORMS

GEORGE S. SHAPIRO

ABSTRACT. Let Λ be a closed and discrete or compact subset of a second countable LCA group G and E a subset of the dual group. Balayage is said to be possible for (Λ, E) if for every finite measure μ on G there is some measure ν on Λ whose Fourier transform, $\hat{\nu}$, agrees on E with $\hat{\mu}$.

If balayage is assumed possible just when μ is a point measure (with the norms of all the measures ν bounded by some constant), then there is a bounded linear projection, B_Λ, from the measures on G onto those on Λ with $(B_\Lambda \mu)\hat{} = \hat{\mu}$ on E. An application is made to balayage in product groups.

Throughout G denotes a second countable locally compact abelian group, Γ its dual group and $\Lambda \subset G$ and $E \subset \Gamma$ closed sets. $M(G)$ is the convolution algebra of finite, regular, complex Borel measures on G and $M(\Lambda)$ is the closed subspace of $M(G)$ consisting of those measures supported by Λ. If $\mu \in M(G)$, $\hat{\mu}$ denotes the (inverse) Fourier transform of μ,

$$\hat{\mu}(\gamma) = \int_G \langle \gamma, x \rangle \, d\mu(x) \quad \text{for } \gamma \in \Gamma.$$

We write $\mu \sim \nu\ (E)$ to mean $\hat{\mu}(\gamma) = \hat{\nu}(\gamma)$ for all $\gamma \in E$.

DEFINITION. Balayage is said to be possible for the pair (Λ, E) if for every $\mu \in M(G)$ there is some $\nu \in M(\Lambda)$ with $\mu \sim \nu\ (E)$.

This situation was introduced and first studied by Beurling ([1], unfortunately unpublished, and [2]) especially for the case where G is the real line. The two cases most studied, and which we deal with here, are those in which Λ is either discrete or compact (in which case E should have the other of these properties to make nontrivial balayage possible).

Our major result (which appeared in the author's dissertation [7]) is that, in the situation considered here, if balayage is possible for (Λ, E), then there is a bounded linear projection of $M(G)$ onto $M(\Lambda)$ (of as small norm as is reasonable) which implements the balayage. Moreover, to obtain this conclusion from a measurable choice argument, we need assume only the possibility of (bounded) balayage of point measures, a fact which is crucial for an application we make to balayage in product groups. In a special case, namely when Λ is discrete and E is a compact convex set with interior in a Euclidean

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space, Beurling [2] showed, by a variational technique, that the existence of a linear balayage operator follows from the possibility of bounded balayage of point measures. His proof also involves integrating a family of measures (somewhat as in Theorem 1) but does not produce a projection.

If balayage is possible for (Λ, E), then an open mapping argument due to Beurling (see [8, §2]) shows the existence of a constant K such that for every $\mu \in M(G)$ there is some $\nu \in M(\Lambda)$ with $\|\mu\| \leqslant K\|\nu\|$ and $\mu \sim \nu\ (E)$. The infimum of constants K for which the above holds is denoted $K(\Lambda, E)$.

We write δ_x for the unit point mass at x so that $\hat{\delta}_x(\gamma) = \langle \gamma, x \rangle$.

THEOREM 1. *Let G be a second countable locally compact abelian group and let $\Lambda \subset G$ be either closed and discrete or compact. Suppose that there is a constant K such that for every $x \in G$ there is a measure $m(x) \in M(\Lambda)$ with $\|m(x)\| \leqslant K$ and $m(x) \sim \delta_x(E)$. Then there is a bounded linear operator B_Λ: $M(G) \to M(\Lambda)$ such that*

1. *$B_\Lambda \mu \sim \mu\ (E)$, for every $\mu \in M(G)$,*
2. *$B_\Lambda \mu = \mu$ for every $\mu \in M(\Lambda)$,*
3. *$\|B_\Lambda\| \leqslant K$.*

In particular, $K(\Lambda, E) \leqslant K$.

PROOF. The gist of the proof is to use a measurable choice result to replace the $m(x)$ by measures $\alpha(x)$ which depend measurably on x and then to set $B_\Lambda \mu = \int_G \alpha(x)\, d\mu(x)$, the integral to be interpreted in the weak-* sense.

We recall first the relevant topological facts.

Let X and Y be Polish spaces (separable spaces whose topologies can be defined by complete metrics). Suppose $Q \subset X \times Y$ is an analytic (Suslin) set whose projection, $\pi_1(Q)$, onto X is all of X. Then there is a function α: $X \to Y$ such that $(x, \alpha(x)) \in Q\ \ \forall x \in X$ and such that $\alpha^{-1}(U)$ is analytic for every open $U \subset Y$. In particular, α is measurable with respect to every regular Borel measure on X. (This is essentially von Neumann's original result [9, Lemma 5] as reformulated over the years. See also [3, Appendix V]. [6, Theorem 3] is a more modern and general formulation.)

Because G is a second countable locally compact group, G is a Polish space.

We set $M_K(\Lambda) = \{\mu \in M(\Lambda) \colon \|\mu\| \leqslant K\}$.

In the case where Λ is discrete (and, hence, countable) we give $M_K(\Lambda)$ its norm topology. Then $M(\Lambda) \simeq l^1(\Lambda)$ is separable so that $M_K(\Lambda)$ is a Polish space. In this case, integrals are really sums but the integral notation is more suggestive and agrees better with the compact case.

In the case where Λ is compact, we give $M_K(\Lambda)$ the restriction of the weak-* topology of $M(\Lambda)$ regarded as the dual of $C(\Lambda)$ (the continuous complex functions on Λ). Since Λ is separable metric, $C(\Lambda)$ is separable and, therefore, the weak-* topology on $M_K(\Lambda)$ is metrizable [4, p. 426]. Since $M_K(\Lambda)$ is also weak-* compact, it is a Polish space.

We deal with both cases together, indicating the requisite variations in the argument as they occur.

We work in $G \times M_K(\Lambda)$. We set

$$S = \{(x, \mu) \in G \times M_K(\Lambda) \colon \mu \sim \delta_x\ (E)\}.$$

S is closed. Indeed, suppose $(x_n, \mu_n) \in S$ for $n = 1, 2, \ldots$ and

$$\lim_{n \to \infty} (x_n, \mu_n) = (x, \mu).$$

Then $\lim_{n\to\infty} \mu_n = \mu$ in $M_K(\Lambda)$ which implies that $\lim_{n\to\infty} \hat{\mu}_n(\gamma) = \hat{\mu}(\gamma)$ for all $\gamma \in \Gamma$. (For the norm topology this is clear; for the weak-* topology it follows because the restriction, $\gamma|\Lambda$, of γ to Λ is in $C(\Lambda)$.) Thus, for $\gamma \in E$,

$$\hat{\mu}(\gamma) = \lim_{n\to\infty} \hat{\mu}_n(\gamma) = \lim_{n\to\infty} \langle \gamma, x_n \rangle = \langle \gamma, x \rangle = \hat{\delta}_x(\gamma).$$

Hence, $(x, \mu) \in S$.

$T = \Lambda \times M_K(\Lambda)$ is closed.

$U = \{(\lambda, \delta_\lambda) \colon \lambda \in \Lambda\}$ is closed. (In the discrete case because $\lambda \neq \lambda'$ implies that $\|\delta_\lambda - \delta_{\lambda'}\| = 2$ and in the compact case because U is the image of the compact set Λ under the map $\lambda \mapsto (\lambda, \delta_\lambda)$.)

We set $Q = S - (T - U)$ (where $-$ denotes relative complementation). Since every closed set in a metric space is a G_δ, it is easily checked that Q is an F_σ and, hence, an analytic set. By construction, Q has the following properties:

(a) $(x, \mu) \in Q \Rightarrow \mu \sim \delta_x\ (E)$.

(b) $(\lambda, \delta_\lambda) \in Q$ for all $\lambda \in \Lambda$.

(c) $(x, m(x)) \in Q$ for all $x \in G - \Lambda$.

(d) $(\lambda, \mu) \in Q$ and $\lambda \in \Lambda \Rightarrow \mu = \delta_\lambda$.

In particular, (b) and (c) imply that $\pi_1(Q) = G$. We now employ the measurable choice result quoted above to obtain a mapping $\alpha \colon G \to M_K(\Lambda)$ such that $(x, \alpha(x)) \in Q$ for all x and such that α is measurable with respect to every $\mu \in M(G)$ (which property we shall henceforth refer to simply as measurability).

Clearly $\|\alpha(x)\| \leqslant K$. From (a) above we deduce that $\alpha(x) \sim \delta_x\ (E)$ for all $x \in G$, while from (d) we see that $\alpha(\lambda) = \delta_\lambda$ if $\lambda \in \Lambda$.

We shall often write $\langle \phi, \mu \rangle$ for $\int \phi \, d\mu$.

$C_0(\Lambda)$ denotes the set of continuous functions on Λ vanishing at infinity ($= C(\Lambda)$ in the compact case). If $\phi \in C_0(\Lambda)$, then the map $\nu \mapsto \langle \phi, \nu \rangle$ is continuous as a map from $M_K(\Lambda)$ (with either topology) to the complex numbers. Thus, $x \mapsto \langle \phi, \alpha(x) \rangle$ is measurable for every such ϕ. This permits us to define, for $\mu \in M(G)$, $B_\Lambda \mu = \int \alpha(x) \, d\mu(x)$, the integral being interpreted in the weak-* sense. More precisely, $B_\Lambda \mu$ is defined by the condition that

$$\langle \phi, B_\Lambda \mu \rangle = \int_G \langle \phi, \alpha(x) \rangle \, d\mu(x) \tag{1}$$

for all $\phi \in C_0(\Lambda)$. Then $B_\Lambda \mu \in M(\Lambda)$ and standard arguments show that B_Λ is linear and that $\|B_\Lambda\| \leqslant K$.

If $\mu \in M(\Lambda)$, then for each $\phi \in C_0(\Lambda)$, we have

$$\langle \phi, B_\Lambda \mu \rangle = \int_\Lambda \langle \phi, \alpha(\lambda) \rangle \, d\mu(\lambda) = \int_\Lambda \langle \phi, \delta_\lambda \rangle \, d\mu(\lambda)$$
$$= \int_\Lambda \phi(\lambda) \, d\mu(\lambda) = \langle \phi, \mu \rangle.$$

Thus, $B_\Lambda \mu = \mu$.

In the compact case, if $\gamma \in E$ then $\gamma|\Lambda \in C_0(\Lambda)$ and

$$(B_\Lambda \mu)\hat{} = \langle \gamma|\Lambda, B_\Lambda \mu \rangle = \int_G \langle \gamma|\Lambda, \alpha(x) \rangle \, d\mu(x)$$
$$= \int_G (\alpha(x))\hat{}\,(\gamma) \, d\mu(x) = \int_G \langle \gamma, x \rangle \, d\mu(x) = \hat{\mu}(\gamma)$$

completing the proof of the theorem in this case.

For the discrete case, we remark that every $\phi \in C(\Lambda)$ can be expressed as a pointwise limit, $\phi = \lim_{n\to\infty} \phi_n$, where $\phi_n \in C_0(\Lambda)$ and $\|\phi_n\|_\infty \leqslant \|\phi\|_\infty$. Then two applications of the dominated convergence theorem show that (1) holds also for $\phi \in C(\Lambda)$. If $\gamma \in E$, then $\gamma|\Lambda \in C(\Lambda)$ and the argument of the previous paragraph now applies in this case also, completing the proof of the theorem.

REMARKS. (1) If Λ is compact, then so is $M_K(\Lambda)$ and now the fact that Q is an F_σ implies that $x \mapsto \alpha(x)$ may be chosen to be Borel measurable. (Use [5, Theorem 1] and the fact that a map between Polish spaces which has a Borel graph is Borel measurable.)

(2) Theorem 1 can be extended, in an ad hoc fashion, to certain other types of Λ, for example, to Λ the disjoint union of a discrete closed set and a compact set. What is required is the existence of a "reasonable" Polish, or at least Suslin, topology $\mathfrak{T}$ on $M_K(\Lambda)$ no weaker than the weak-* topology and such that $\mathfrak{T}$-convergence implies pointwise convergence of Fourier transforms on E. For general closed Λ, however, $M_K(\Lambda)$ will *not* be second countable.

THEOREM 2. *Let G_1 and G_2 be second countable LCA groups with dual groups Γ_1 and Γ_2. Regard $\Gamma_1 \times \Gamma_2$ as the dual of $G_1 \times G_2$. Suppose $\Lambda_i \subset G_i$ $(i = 1, 2)$ are either both closed and discrete or both compact and let $E_i \subset \Gamma_i$. If balayage is possible for (Λ_i, E_i), $i = 1, 2$, then balayage is possible for $(\Lambda_1 \times \Lambda_2, E_1 \times E_2)$ and*

$$K(\Lambda_1 \times \Lambda_2, E_1 \times E_2) \leqslant K(\Lambda_1, E_1) K(\Lambda_2, E_2).$$

PROOF. Let $\varepsilon > 0$ and suppose $(x_1, x_2) \in G_1 \times G_2$. Then, for $i = 1, 2$, there is some $\mu_i \in M(\Lambda_i)$ with $\|\mu_i\| \leqslant K(\Lambda_i, E_i) + \varepsilon$ and $\mu_i \sim \delta_{x_i}(E_i)$. Let $\mu = \mu_1 \times \mu_2$ be the product measure. Then $\|\mu\| \leqslant \|\mu_1\| \, \|\mu_2\|$. If $(\gamma_1, \gamma_2) \in E_1 \times E_2$, a simple computation gives

$$\hat{\mu}(\gamma_1, \gamma_2) = \hat{\mu}_1(\gamma_1)\hat{\mu}_2(\gamma_2) = \langle \gamma_1, x_1 \rangle \langle \gamma_2, x_2 \rangle$$
$$= \langle (\gamma_1, \gamma_2), (x_1, x_2) \rangle.$$

Thus, $\mu \sim \delta_{(x_1, x_2)} (E_1 \times E_2)$.

Since $\Lambda_1 \times \Lambda_2$ is either closed and discrete or else compact, Theorem 1 now implies that

$$K(\Lambda_1 \times \Lambda_2, E_1 \times E_2) \leqslant (K(\Lambda_1, E_1) + \varepsilon)(K(\Lambda_2, E_2) + \varepsilon).$$

Since ε is arbitrary, the result follows.

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Department of Mathematics, City University of New York, Brooklyn College, Brooklyn, New York 11210