PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 2, December 1976

UNBOUNDED DERIVATIONS OF GROUP C^*- ALGEBRAS

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ABSTRACT. We describe a construction which yields unbounded derivations and strongly continuous one-parameter automorphism groups of certain group C^*-algebras. As an application, we show that a simple C^*-algebra can have an automorphism group which is not approximately inner.

1. A strongly continuous one-parameter automorphism group of a C^*-algebra $\mathfrak{A}$ is a continuous homomorphism $t \mapsto \alpha_t$ from the real numbers into the group of *-automorphisms of $\mathfrak{A}$ equipped with the topology of simple norm convergence. We say that (α_t) is approximately inner if there is a sequence (H_n) of selfadjoint elements of $\mathfrak{A}$ such that $\|e^{itH_n}Ae^{-itH_n} - \alpha_t(A)\| \to 0$ uniformly on compact subsets of $\mathbf{R}$, for each A in $\mathfrak{A}$. Strongly continuous one-parameter automorphism groups of UHF algebras have been investigated in a recent series of papers by Powers and Sakai [5]–[8], who conjecture that such groups are always approximately inner.

In this paper we show how to construct strongly continuous automorphism groups in certain group C^*-algebras which are not approximately inner. In particular, the construction works when the group is the free group on two generators. The reduced C^*-algebra of this group is simple [4], so our results show that a simple C^*-algebra with identity can, in general, have strongly continuous one-parameter automorphism groups which are not approximately inner, in contrast to the Powers-Sakai conjecture for the UHF case.

We should like to thank R. Powers and S. Sakai for supplying us with preprints of their work.

2. Let G be a (discrete) group. For h in G let ε_h be the characteristic function of $\{h\}$, so that $\{\varepsilon_h: h \in G\}$ is an orthonormal basis for $l^2(G)$. For g in G define a unitary operator U_g on $l^2(G)$ by $U_g\varepsilon_h = \varepsilon_{gh}$. Let $\mathfrak{A}_0$ be the set of all finite linear combinations of elements of $\{U_g: g \in G\}$. The reduced C^*-algebra $C_r^*(G)$ of G is the norm closure of $\mathfrak{A}_0$ in the algebra of all bounded operators on $l^2(G)$. The (full) C^*-algebra $C^*(G)$ of G is the enveloping C^*-algebra (see §2.7 of [2]) of the convolution group algebra $l^1(G)$.

Define a state ω of $C_r^*(G)$ by

Received by the editors March 16, 1976.
AMS (MOS) subject classifications (1970). Primary 46L05.
[1] Research supported by a grant from MOSOI, Iran.

$$\omega(A) = \langle A\varepsilon_e, \varepsilon_e \rangle,$$

where e is the identity element of G. If $A = \sum \gamma_g U_g \in \mathfrak{A}_0$, then $\omega(A) = \gamma_e$.

Let λ be a homomorphism from G to the additive group of real numbers, and define a linear operator δ_λ on $\mathfrak{A}_0$ by

$$\delta_\lambda(\sum \gamma_g U_g) = \sum i\lambda(g)\gamma_g U_g.$$

For g, h in G we have

$$\begin{aligned}\delta_\lambda(U_g U_h) = \delta_\lambda(U_{gh}) = i\lambda(gh)U_g U_h &= i\lambda(g)U_g U_h + i\lambda(h)U_g U_h \\ &= \delta_\lambda(U_g)U_h + U_g\delta_\lambda(U_h),\end{aligned}$$

from which it follows that $\delta_\lambda(AB) = \delta_\lambda(A)B + A\delta_\lambda(B)$ for all A, B in $\mathfrak{A}_0$. Also

$$\delta_\lambda(U_g^*) = \delta_\lambda(U_{g^{-1}}) = i\lambda(g^{-1})U_{g^{-1}} = -i\lambda(g)U_g^* = \delta_\lambda(U_g)^*,$$

so that $\delta_\lambda(A^*) = \delta_\lambda(A)^*$ for all A in $\mathfrak{A}_0$. Thus δ_λ is a derivation of $C_r^*(G)$. (By a derivation δ of a C^*-algebra $\mathfrak{A}$ we mean a linear mapping from a dense $*$-subalgebra $\mathfrak{D}(\delta)$ of $\mathfrak{A}$ into $\mathfrak{A}$ such that

$$\begin{aligned}\delta(AB) &= \delta(A)B + A\delta(B) \qquad (A, B \in \mathfrak{D}(\delta)),\\ \delta(A^*) &= \delta(A)^* \qquad (A \in \mathfrak{D}(\delta)).\end{aligned}$$

In this we differ slightly from the usage of Bratteli and Robinson [1], who take δ to be skew-adjoint rather than selfadjoint.)

THEOREM 1. *For λ in* Hom $(G, \mathbf{R})$, *δ_λ is a closable derivation of $C_r^*(G)$ whose closure $\bar{\delta}_\lambda$ is the infinitesimal generator of a strongly continuous one-parameter group of automorphisms (α_t^λ) of $C_r^*(G)$. If $\lambda \neq 0$ then δ_λ is unbounded and (α_t^λ) is not approximately inner.*

PROOF. For $A = \sum \gamma_g U_g$ in $\mathfrak{A}_0$, we have $\omega(\delta_\lambda(A)) = i\lambda(e)\gamma_e = 0$. For g in G the power series

$$\sum_{n \geqslant 0} \frac{z^n}{n!}\delta_\lambda^n(U_g) = \sum_{n \geqslant 0} \frac{(i\lambda(g)z)^n}{n!} U_g$$

has nonzero (in fact, infinite!) radius of convergence, so that $\mathfrak{A}_0$ is a dense set of analytic elements for δ_λ. It follows from Theorem 4 of [1] that δ_λ is closable and that $\bar{\delta}_\lambda$ is the infinitesimal generator of a strongly continuous one-parameter automorphism group (α_t^λ), as required.

Suppose now that $\lambda \neq 0$, so that $\lambda(g) > 0$ for some g in G. For any positive integer n we have $\delta_\lambda(U_{g^n}) = in\lambda(g)U_{g^n}$, from which it is clear that δ_λ is unbounded. If $A = U_g^*$ then

$$-iA^* \delta_\lambda(A) = \lambda(g^{-1}) U_g U_g^* = -\lambda(g) I.$$

If (α_t^λ) were approximately inner, then by Theorem 2.3 of [5] there would be a ground state ρ for (α_t^λ). But by Theorem 2.2 of [5] such a state has the property that $-i\rho(A^* \delta_\lambda(A)) \geqslant 0$ $(A \in \mathfrak{A}_0)$. This is clearly impossible for $A = U_g^*$, so we conclude that (α_t^λ) is not approximately inner.

Notice that the automorphisms α_t^λ are induced by the unitary group on $l^2(G)$ given by $\varepsilon_h \mapsto e^{it\lambda(h)} \varepsilon_h$. Thus $\mathfrak{A}_0$ is invariant under α_t^λ $(t \in \mathbf{R})$.

3. We now prove the analogous result to Theorem 1 for the full group C^*-algebra $C^*(G)$. We shall need the following lemma, which is a consequence of Theorem 1 of [1] (see also Theorem 4.1 of [5]).

LEMMA 2. *Let δ be a derivation of a C^*-algebra $\mathfrak{A}$ with domain $\mathfrak{D}(\delta)$ such that some extension $\hat{\delta}$ of δ is the infinitesimal generator of a strongly continuous one-parameter group of automorphisms of $\mathfrak{A}$. Suppose that the two sets $\{\delta(A) \pm A : A \in \mathfrak{D}(\delta)\}$ are dense in $\mathfrak{A}$. Then $\hat{\delta}$ is the closure of δ.*

PROOF. Since $\hat{\delta}$ is closed, δ is certainly closable. By Theorem 1 of [1],

$$\|\hat{\delta}(A) - rA\| \geqslant |r|\,\|A\| \qquad (r \in \mathbf{R}, A \in \mathfrak{D}(\hat{\delta})).$$

For any B in $\mathfrak{A}$ we can find a sequence (A_n) in $\mathfrak{D}(\delta)$ such that $\delta(A_n) + A_n \to B$. Since $\|A_m - A_n\| \leqslant \|\delta(A_m - A_n) + (A_m - A_n)\|$, (A_n) is a Cauchy sequence with limit A, say. Then $\delta(A_n) \to B - A$, so $A \in \mathfrak{D}(\bar{\delta})$, $\bar{\delta}(A) = B - A$ and thus $B = \bar{\delta}(A) + A$. Hence we have

$$\|\delta(A) - rA\| \geqslant |r|\,\|A\| \qquad (r \in \mathbf{R}, A \in \mathfrak{D}(\bar{\delta})),$$

$$\{\bar{\delta}(A) + A : A \in \mathfrak{D}(\bar{\delta})\} = \mathfrak{A}$$

and (similarly) $\{\bar{\delta}(A) - A : A \in \mathfrak{D}(\bar{\delta})\} = \mathfrak{A}$.

It follows from Theorem 1 of [1] that $\bar{\delta}$ is the infinitesimal generator of a strongly continuous one-parameter group of automorphisms of $\mathfrak{A}$, which must be the same as that generated by $\hat{\delta}$, so that $\bar{\delta} = \hat{\delta}$.

THEOREM 3. *For λ in* Hom $(G, \mathbf{R})$, *δ_λ is a closable derivation of $C^*(G)$ whose closure $\hat{\delta}_\lambda$ is the infinitesimal generator of a strongly continuous one-parameter group of automorphisms $(\hat{\alpha}_t^\lambda)$ of $C^*(G)$. If $\lambda \neq 0$ then δ_λ is unbounded and $(\hat{\alpha}_t^\lambda)$ is not approximately inner.*

PROOF. As noted at the end of §2, $\mathfrak{A}_0$ is invariant under the automorphisms α_t^λ. If p is any C^*-seminorm on $\mathfrak{A}_0$ then the composite mapping $p\alpha_t^\lambda$ is also a C^*-seminorm. The $C^*(G)$-norm on $\mathfrak{A}_0$ is, by definition, the supremum of all such C^*-seminorms, and it follows that each of the automorphisms α_t^λ is isometric for this norm. Thus (α_t^λ) extends by continuity to a strongly continuous one-parameter automorphism group $(\hat{\alpha}_t^\lambda)$ of $C^*(G)$, whose infinitesimal generator $\hat{\delta}_\lambda$ is clearly an extension of δ_λ. For $A = \sum \gamma_g U_g$ in $\mathfrak{A}_0$, we have

$$\delta_\lambda(A) \pm A = \sum (i\lambda(g) \pm 1)\gamma_g U_g.$$

Since $i\lambda(g) \pm 1$ cannot be zero it is clear that the range of $\delta_\lambda \pm 1$ is the whole of $\mathfrak{A}_0$, and it follows from Lemma 2 that δ_λ is closable, with closure $\hat{\delta}_\lambda$.

We already know that δ_λ is unbounded if $\lambda \neq 0$. To see that $(\hat{\alpha}_t^\lambda)$ is not approximately inner, let π denote the canonical projection (i.e. the quotient map) from $C^*(G)$ onto $C_r^*(G)$. Then the restriction of π to $\mathfrak{A}_0$ is the identity map, and $\pi\hat{\alpha}_t^\lambda = \alpha_t^\lambda\pi$. Suppose there is a sequence (H_n) of selfadjoint elements of $C^*(G)$ with $\|e^{itH_n}Ae^{-itH_n} - \hat{\alpha}_t^\lambda(A)\| \to 0$ $(A \in C^*(G))$. Then $\|e^{it\pi(H_n)}\pi(A)e^{-it\pi(H_n)} - \alpha_t^\lambda(\pi(A))\| \longrightarrow 0$, and it would follow that (α_t^λ) was approximately inner, contradicting Theorem 1. Thus $(\hat{\alpha}_t^\lambda)$ is not approximately inner.

4. When is Hom $(G, \mathbf{R}) \neq (0)$? To answer this question, we observe first that, since $\mathbf{R}$ is abelian, the commutator subgroup $[G, G]$ of G must be contained in the kernel of every element of Hom $(G, \mathbf{R})$ and so there is a natural surjection Hom $(G, \mathbf{R}) \mapsto$ Hom $(G/[G, G], \mathbf{R})$. Thus the problem is reduced to the case where G is abelian.

PROPOSITION 4. Hom $(G, \mathbf{R}) = 0$ *if and only if* $G/[G, G]$ *is a torsion group.*

PROOF. As explained above, we may replace G by $G/[G, G]$ and suppose that G is abelian. If G is a torsion group then (since $\mathbf{R}$ is torsion-free) it is clear that Hom $(G, \mathbf{R}) = (0)$. Suppose that G is not a torsion group, so that there is an element h of G of infinite order. Then h freely generates an infinite cyclic subgroup H of G, and we can certainly choose a nonzero homomorphism λ from H into $\mathbf{R}$. However, $\mathbf{R}$ is an injective object in the category of abelian groups (since it is a divisible group) and so λ extends to a nonzero homomorphism from G to R. Thus Hom $(G, \mathbf{R}) \neq (0)$.

5. **Comments and examples.** Consider first the case where G is abelian. If we regard the element $\sum \gamma_g U_g$ of $\mathfrak{A}_0$ as the Fourier series of the function $\chi \mapsto \sum \gamma_g \chi(g)$ $(\chi \in \hat{G})$ on the compact dual group $\hat{G}$, acting by pointwise multiplication as a bounded operator on $L^2(\hat{G})$, then we can easily verify that the group algebras $C_r^*(G)$ and $C^*(G)$ are both isomorphic to the C^*-algebra $C(\hat{G})$ of all continuous complex-valued functions on $\hat{G}$. (In general, $C_r^*(G)$ and $C^*(G)$ are isomorphic if and only if G is amenable–see [3].) If $G = \mathbf{Z}$ (the additive group of integers) and λ is the inclusion map from $\mathbf{Z}$ into $\mathbf{R}$ then, under the isomorphism between $C^*(\mathbf{Z})$ and $C(\Gamma)$ (where Γ is the circle group), α_t^λ corresponds to the automorphism given by a rotation of Γ through $2\pi t$ and δ_λ corresponds (up to a scalar multiple) with ordinary differentiation of a function in $C(\Gamma)$.

For our second example we take G to be the free group on two generators. In this case it is clear that Hom $(G, \mathbf{R}) \neq (0)$. It has been shown by Powers [4] that $C_r^*(G)$ (which is not isomorphic to $C^*(G)$) is simple. By Theorem 1,

$C_r^*(G)$ has strongly continuous one-parameter groups of automorphisms which are not approximately inner.

We remark in conclusion that our results do not cast any doubt on the validity of the Powers-Sakai conjecture, since a UHF algebra can never be the group C^*-algebra (reduced or full) of a discrete group.

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