PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 2, December 1976

BANACH SPACES ON WHICH NONTRIVIAL SAKS SPACES EXIST

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ABSTRACT. A necessary and sufficient condition for a Banach space to have on itself a nontrivial Saks space is given.

Let X be a vector space, $\| \|$ a homogeneous norm on X, and $\| \|^*$ an F-norm on X. The closed unit ball X_s in the normed space $(X, \| \|)$ endowed with the metric induced by $\| \|^*$ is said to be Saks set and is denoted $X_s(\| \|, \| \|^*)$. If a Saks set is complete, it is called Saks space. In the sequel the norm $\| \|$ is called the fundamental one, and $\| \|^*$ the starred one. A sequence $(x_n) \subset X$ is said to be ω-convergent iff (x_n) is bounded in $(X, \| \|)$ and convergent in $(X, \| \|^*)$. If $X_s(\| \|, \| \|^*)$ is a Saks space then the fundamental norm is stronger (= stronger or equivalent) than the starred one iff $(X, \| \|)$ is a Banach space [7, II B]. While dealing with general Saks spaces another situation can also occur. This is exhibited by the following example:

Let $X = L^1(-\infty, \infty) \cap L^\infty(-\infty, \infty)$. Take the L^∞-norm as the fundamental one, and the L^1-norm as the starred one. The space X with each of these norms is an incomplete normed space, but the corresponding Saks set $X_s(\| \|, \| \|^*)$ is complete and thus forms a Saks space.

On the other hand, the case when the starred norm is stronger than the fundamental one is not interesting since then ω-convergence is simply convergence in $(X, \| \|^*)$ (an F-space if $X_s(\| \|, \| \|^*)$ is a Saks space). In this case $X_s(\| \|, \| \|^*)$ will be called the trivial one. Note that, with this terminology, a Saks space $X_s(\| \|, \| \|^*)$ on a Banach space $(X, \| \|)$ is trivial iff $\| \|$ and $\| \|^*$ are equivalent in X.

Let $X_s(\| \|, \| \|^*)$ be a given Saks space. In the "theory of Saks spaces" one usually assumes $X_s(\| \|, \| \|^*)$ to satisfy some additional conditions. We now formulate two of such conditions.

(Σ_1) Given any $x_0 \in X_s$ and $r > 0$ there exists $d > 0$ such that every element $x \in X_s$, for which $\|x\|^* < d$, can be written in the form $x = x_1 - x_2$, with $x_1, x_2 \in K(x_0, r)$ (where $K(x_0, r)$ denotes the open ball with centre x_0 and radius r in the space $X_s(\| \|, \| \|^*)$).

Received by the editors March 5, 1974 and, in revised form, February 11, 1975.

AMS (MOS) subject classifications (1970). Primary 46A15, 46B99; Secondary 47B99.

Key words and phrases. Banach space, Saks space, linear mapping on a Saks space, two-norm space, summability theory.

Let $(x_n) \subset X_s(\| \ \|, \| \ \|^*)$, $x_n \to 0$ and $\varepsilon_n > 0$, $\varepsilon_n \to 0$ be given. Then there exist an increasing subsequence (n_k) of indices and a sequence $(\bar{x}_{n_k}) \subset X_s$ such that

(i) $$\|x_{n_k} - \bar{x}_{n_k}\|^* < \varepsilon_{n_k} \quad \text{for } k = 1, 2, \ldots,$$

(Σ_2) (ii) $$\sum_{k=1}^{m} \lambda_k \bar{x}_{n_k} \in X_s, \quad \text{where } m = 1, 2, \ldots, \quad \lambda_k = 0 \text{ or } 1,$$

(iii) for each sequence $\lambda = (\lambda_i)$ composed of 0's and 1's there exists an element $\bar{x}_\lambda \in X_s$ such that

$$\sum_{k=1}^{i} \lambda_k \bar{x}_{n_k} \to \bar{x}_\lambda \ (\text{in } \| \ \|^*) \text{ as } i \to \infty.$$[1]

For some comments concerning these conditions see [7], [4], [5].

A linear map on a Saks set is by definition the restriction of a linear map on X.

The condition (Σ_1) and also (Σ_2) imply the so called (BS) property: Let ξ_n be linear continuous functionals on $X_s(\| \ \|, \| \ \|^*)$. The Saks set $X_s(\| \ \|, \| \ \|^*)$ is said to have the (BS) property iff the following condition is fulfilled:

(BS) $\xi_n(x) \to 0$ for $x \in X_s$ implies that ξ_n are equicontinuous at 0, i.e., $x_n \to 0$ in $X_s(\| \ \|, \| \ \|^*)$ implies $\xi_n(x_n) \to 0$.

Saks spaces having the (BS) property were investigated in [8], [10] and [5].

Saks spaces may look–at first glance–rather strange objects, but their general theory was studied quite extensively [4]–[10], as well as the related theory of so-called two-norm spaces. A number of applications, as e.g. to summability theory [1], [2], [9], [11], [12], [13], [15], [16], has proved this theory to be useful. However, the importance of every theory depends essentially on how many objects can be covered by it. It thus seems rather surprising that, as far as I know, the problem–on which normed nontrivial Saks spaces can exist–was posed explicitly only recently [5]. No satisfactory answer to this problem was obtained therein. We quote here the following observation made there:

"Every "good" theorem on the continuity of linear mappings from a Saks space into a normed space turns out to minimize, via the identity operator, the possible number of nontrivial Saks spaces."

According to this point of view we first prove a theorem on the continuity of linear mappings. We denote by l^∞ the space of all bounded sequences with the supremum norm, by $e^{(n)}$, $n = 1,2,\ldots,$ the sequence $(0, \ldots, 0,1,0, \ldots)$ with the one in the nth position. If on a vector space two linear topologies α and β are given, then we say that β is α-polar (resp. sequentially α-polar) if β has a base of α-closed (resp. sequentially α-closed) neighbourhoods at 0.

THEOREM 1. *Let $X_s(\| \ \|, \| \ \|^*)$ be a Saks space satisfying the (Σ_2) condition,*

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[1] If $X_s(\| \ \|, \| \ \|^*)$ is a Saks space condition (iii) is superfluous. In fact, we can choose such a subsequence (n_k) that $\Sigma_{k=1}^{\infty}\|\bar{x}_{n_k}\|^*$ is convergent. Hence by completeness $\Sigma_{k=1}^{\infty}\bar{x}_{n_k}$ is absolutely summable in $X_s(\| \ \|, \| \ \|^*)$.

(Y, α) a Hausdorff topological vector space, β a sequentially α-polar locally convex Hausdorff topology on Y, $A\colon X_s(\|\ \|, \|\ \|^*) \to (Y, \alpha)$ a continuous linear mapping. If $A\colon X_s(\|\ \|, \|\ \|^*) \to (Y, \beta)$ is not continuous then (Y, β) contains a subspace linearly homeomorphic to l^∞.

The proof of this theorem, as is easily seen, is similar to that of Bennett and Kalton given for Theorem 1 in [3].

PROOF. By Eidelheit-Mazur theorem we can assume the F-norm $\|\ \|^*$ to be monotone. If the mapping A is not β-continuous then we find, using the (Σ_2) condition (cf. [4, 1.1] and [7, I, p. 270]), a sequence $(\bar{x}_n) \subset X_s$ such that the partial sums of the series $\Sigma_{n=1}^\infty \bar{x}_n$ belong to X_s and $\Sigma_{n=1}^\infty \|\bar{x}_n\|^* < \infty$; we find also a continuous seminorm p on (Y, β) such that $p(A(\bar{x}_n)) \geqslant 1$ for all n. $X_s(\|\ \|, \|\ \|^*)$ being complete and $\|\ \|^*$ monotone, $\Sigma_{n=1}^\infty a_n\bar{x}_n$ converges uniformly in $X_s(\|\ \|, \|\ \|^*)$ for all sequences (a_n) such that $|a_n| \leqslant 1$ for all n. Define $V\colon l^\infty \to (Y, \alpha)$ by $V(a) = \Sigma_{n=1}^\infty A(a_n\bar{x}_n)$; V is continuous. Using Baire's theorem and α-polarity of β one easily infers that $V\colon l^\infty \to (Y, \beta)$ is bounded, hence, continuous. As $p(A(\bar{x}_n)) \geqslant 1$ for all n, we can find a sequence (ξ_n) of equicontinuous linear functionals on (Y, β) such that $|\xi_n(\bar{x}_n)| \geqslant 1$. Define further $R\colon (Y, \beta) \to l^\infty$ by $R(y) = (\xi_n(y))$; R is continuous. Consider $RV\colon l^\infty \to l^\infty$; we have $\|RV(e^{(n)})\| \geqslant 1$ for all n. Hence, by the Orlicz-Pettis theorem, $\Sigma_{n=1}^\infty RV(e^{(n)})$ (see e.g. [3]) and it follows that RV is not weakly compact. By Corollary 1.4 of Rosenthal [14], RV is an isomorphism on some subspace H of l^∞ which is itself isomorphic to l^∞. Hence $V(H)$ is isomorphic to l^∞ and (Y, β) contains a subspace isomorphic to l^∞.

With the same notation:

COROLLARY 2. *Let $A\colon X_s(\|\ \|, \|\ \|^*) \to (Y, \alpha)$ be a continuous linear map. If (Y, β) does not contain any isomorphic copy of l^∞ then $A\colon X_s(\|\ \|, \|\ \|^*) \to (Y, \beta)$ is continuous.*

We have also

COROLLARY 3. *Let $X_s(\|\ \|, \|\ \|^*)$ be a Saks space satisfying the (Σ_2) condition. If $(X, \|\ \|)$ does not contain any isomorphic copy of L^∞ then $X_s(\|\ \|, \|\ \|^*)$ is trivial (i.e., $\|\ \|^*$ is stronger than $\|\ \|$ in X).*

In fact, take A = the identity from $X_s(\|\ \|, \|\ \|^*)$ into $(X, \|\ \|)$. $X_s(\|\ \|, \|\ \|^*)$ being complete, the unit ball of $(X, \|\ \|)$ is closed in $(X, \|\ \|^*)$; thus the topology of the fundamental norm is polar with respect to the topology of the starred norm. Hence the identity $I\colon X_s(\|\ \|, \|\ \|^*) \to (X, \|\ \|)$ is continuous, i.e., $\|\ \| \leqslant \|\ \|^*$ on X_s. Consequently, there exists a balanced $\|\ \|^*$-neighbourhood of zero, U say, such that $U \cap X_s \subset \frac{1}{2} X_s$. Clearly no point in U can be outside $\frac{1}{2} X_s$; hence, $U \subset X_s$. Thus for each $\varepsilon > 0$, $\varepsilon U \subset \varepsilon X_s$. This proves the result.

We arrive at the main theorem of the present paper.

THEOREM 4. *Let $(X, \|\ \|)$ be a Banach space. The following are equivalent.*
(i) *There exists on X a nontrivial Saks space satisfying the (Σ_2) condition.*
(ii) *$(X, \|\ \|)$ contains a subspace linearly homeomorphic to l^∞.*

PROOF. In view of what has been proved above only the implication

(ii) $\Rightarrow$ (i) needs the proof. If (ii) is satisfied then (denoting $(X, \| \|) = X$) $X = Z \oplus l^\infty$. In the sequel the generic notation for elements of Banach spaces X, Z, l^∞ is x, z, y, respectively, $\| \|_\infty$ denotes the supremum norm of l^∞, $\| \|_\infty^*$ the pointwise convergence B_0-norm in l^∞. So we have $z = P_1(x)$, $y = P_2(x)$ where P_1, P_2 are first and second projections, respectively, and $(X, \| \|)$ is isomorphic with $(X, \| \|_1)$ where $\| \|_1$ is defined by $\|x\|_1 = \max(\|z\|, \|y\|_\infty)$. Define $\| \|^*$ by $\|x\|^* = \max(\|z\|_1, \|y\|_\infty^*)$. Consider the Saks set $X_s(\| \|_1, \| \|^*) = X_s$. It is clearly nontrivial. If $(x_n) \subset X_s$ is Cauchy then

$$\begin{aligned}\lim\|x_n - x_m\|^* &= \lim\|z_n - z_m + y_n - y_m\|^* \\ &= \lim \max(\|z_n - z_m\|_1, \|y_n - y_m\|_\infty^*) \longrightarrow 0 \qquad (n, m \longrightarrow \infty).\end{aligned}$$

Hence (z_n) is Cauchy, also $(y_n) \subset l_s^\infty$ [2] and is Cauchy therein. Consequently both are convergent which proves that X_s is a Saks space. That X_s satisfies the (Σ_2) condition will be proved below.

The following problems arise:

What is the full characterization of those Banach spaces which contain a nontrivial Saks space fulfilling the (Σ_1) condition (or having the (BS)property)?

What is the relation between the (Σ_1) and (Σ_2) conditions?

Our characterization of Banach spaces which contain nontrivial Saks spaces satisfying the (Σ_2) condition is practically sufficient since all known concrete examples of Saks spaces satisfy both conditions (or none of them). We shall show that (ii) is sufficient for a Banach space to have a nontrivial Saks space satisfying the (Σ_1) condition (and, a fortiori, having the (BS) property). In fact, we have, with the notation used in the proof above,

PROPOSITION 5. *X_s satisfies the (Σ_1) and (Σ_2) conditions.*

PROOF. Denote

D_1 = the set of interior points of the unit ball in $(Z, \| \|_1)$.

D_2 = the set of those points in l_s^∞ in which the "(Σ_1) condition at a point" [7, I, 1.32] is satisfied.

We recall that the condition (Σ_1) is satisfied at the point $x_0 \in X_s$ if there exists $d > 0$ such that $\|x\|^* < d$ implies $x_0 + x \in X_s$ for every $x \in X_s$.

(*) If the condition (Σ_1) is satisfied at any point of a set dense in a Saks space then this Saks space satisfies conditions (Σ_1) and (Σ_2) [7, loc. cit.].

Consider the points in $D = \{x \in X_s,\ x = z + y\colon z \in D_1, y \in D_2\}$. Take $x_0 \in D$ and x such that $x \in X_s$, $\|x\|^* < d$; then $y \in l_s^\infty$. One may take d so small that $y_0 + y \in l_s^\infty$ (by the definition of D_2) and $z_0 + z \in D_1$ (by the definition of D_1). Hence $x_0 + x \in X_s$, i.e., the condition (Σ_1) is satisfied at any point in D. By [7, I, 1.52(I)] the set D_2 is dense in l_s^∞; thus D is dense in X_s. Hence the proposition follows by (*).

REMARK. It seems unlikely that the class of Banach spaces containing nontrivial Saks spaces which satisfy (Σ_2) may be larger than that one

[2] We denote by l_s^∞ the Saks set $l_s^\infty(\| \|_\infty, \| \|_\infty^*)$. It is a Saks space satisfying the (Σ_1) and (Σ_2) conditions [7, I].

containing nontrivial Saks spaces which satisfy (Σ_1), since (Σ_2) seems to assure "better" theorems concerning the continuity of linear operators on Saks spaces. So, if both classes do not coincide, a Saks space satisfying (Σ_1) but not (Σ_2) should be found.

POSTSCRIPTUM. In the meantime the results of this paper were strengthened in [17].

REFERENCES

1. A. Alexiewicz and W. Orlicz, *On summability of double sequences*. I, II, Ann. Polon. Math. **2**(1955), 170–181; ibid **6**(1959), 171–180. MR**18**, 205; **21** #6488.

2. ______ , *Consistency theorems for Banach space analogues of Toeplitzian methods of summability*, Studia Math. **18**(1959), 199–210. MR**21** #7382.

3. G. Bennett and N. J. Kalton, *Addendum to "FK-spaces containing c_0"*, Duke Math. J. **39**(1972), 819–821. MR**47** #2312.

4. I. Labuda, *Continuity of operators on Saks spaces*, Studia Math. **51**(1974), 11–21.

5. I. Labuda and W. Orlicz, *Some remarks on Saks spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **22**(1974), 909–914.

6. W. Matuszewska and W. Orlicz, *On a class of Saks spaces*, Bull. Acad. Polon. Sci. Cl. III **5**(1957), 611–614. MR**19**, 564.

7. W. Orlicz, *Linear operations in Saks spaces*. I, II, Studia Math. **11**(1950), 237–272; ibid. **15**(1955), 1–25. MR **12**, 418; **17**, 511.

8. W. Orlicz and V. Ptak, *Some remarks on Saks spaces*, Studia Math. **16**(1957), 56–68. MR**20** #1198.

9. W. Orlicz, *On the continuity of linear operations in Saks spaces with an application to the theory of summability*, Studia Math. **16**(1957), 69–73. MR**20** #1135.

10. ______ , *Contribution to the theory of Saks spaces*, Fund. Math. **44**(1957), 270–294. MR**22** #4938.

11. ______ , *Funktionalanalysis und allgemeine Theorie der linearen Transformationen*, Colloq. sur la théorie des suites (Bruxelles, 1957), Centre Belge de Recherches Mathématiques, Librairie Gauthier-Villars, Paris, 1958, pp. 131–147. MR**21** #1469.

12. ______ , *On the summability of bounded sequences by continuous methods*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **6**(1958), 549–556. MR**21** #1470.

13. A. Persson, *Summation methods on locally compact spaces*, Thesis, University of Lund, 1965.

14. H. P. Rosenthal, *On relatively disjoint families of measures, with some applications to Banach space theory*, Studia Math. **37**(1970), 13–36. MR**42** #5015.

15. J. J. Sember, *Variational FK spaces and two-norm convergence*, Math. Z. **119**(1971), 153–159. MR**43** #6627.

16. A. Wilansky, *Topics in functional analysis*, Lecture Notes in Math., no. 45, Springer-Verlag, Berlin and New York, 1967. MR**36** #6901.

17. I. Labuda, *On the existence of non-trivial Saks sets and continuity of linear mappings acting on them*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 23 (1975), 885–890.

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