ROTUNDITY AND SMOOTHNESS IN CONJUGATE SPACES

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Abstract. It is shown that the James space J admits an equivalent norm such that its third conjugate space is rotund.

A Banach space $B$ is said to be rotund if $\|x + y\| < 2$ whenever $x, y \in S(B) \equiv \{z \in B: \|z\| = 1\}$ and $x \neq y$. A Banach space $B$ with conjugate space $B^*$ is said to be smooth if for every $x \in S(B)$ there exists a unique $x^* \in S(B^*)$ such that $x^*(x) = 1$. Note that $B$ is smooth if $B^*$ is rotund, and $B$ is rotund if $B^*$ is smooth.

The purpose of this note is to show the existence of a nonreflexive Banach space with rotund third conjugate space, a matter of interest raised by Rainwater [8] and Diestel [3, p. 43]. Specifically, we show that the quasi-reflexive space $J$ of James [6] admits an equivalent norm such that its third conjugate space is rotund.

For a Banach space $B$, we denote its successive conjugate spaces by $B^*, B^{**}, B^{***}$, and $B^{(4)}$. We denote the canonical isometric embeddings of $B, B^*, B^{**}$ into $B^{**}, B^{***}, B^{(4)}$ by $Q_0, Q_1, Q_2$ respectively. When no ambiguity exists, we shall omit these embeddings and write, for example, $x \in B^{**}$ instead of $Q_0(x) \in B^{**}$ for $x \in B$. Recall that we may write $B^{***} = B^* \oplus B^\perp$, and that $\|x^*\| \leq \|x^* + x^\perp\|$ for all $x^* \in B^*$ and $x^\perp \in B^\perp$.

For an arbitrary nonreflexive Banach space $B$, Dixmier [4] has shown that $B^{(4)}$ is not rotund. Phelps, Day [1, p. 70], Giles [5], and Rainwater [8] have independently strengthened Dixmier's result by showing that $B^{***}$ is not smooth for $B$ nonreflexive. Combining these results, we have that if $B$ is nonreflexive, then the $n$th conjugate space $B^{(n)}$, for $n > 4$, is neither rotund nor smooth. On the positive side, from the work of Zizler [11], we have that if $B^*$ is separable, then $B$ admits an equivalent norm such that $B$ and $B^*$ are both rotund and smooth, and $B^{**}$ is rotund. In [9], we established the existence of a smooth, nonreflexive second conjugate space. In this paper, we complete the picture of rotundity and smoothness in conjugate spaces by showing the existence of a rotund, nonreflexive third conjugate space. Combining the proof of our result with a result of Zizler [11, p. 200], we have the best possible result that there exists a nonreflexive Banach space $B$ such that

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B, $B^*$, and $B^{**}$ are both rotund and smooth, and $B^{***}$ is rotund.

We shall need the following strengthenings of rotundity.

A Banach space $B$ is said to be locally uniformly rotund if whenever $x \in S(B)$ and $\{x_n\}$ is a sequence in $B$ such that $\|x_n\| \to 1$ and $\|x + x_n\| \to 2$, then $x_n \to x$.

A conjugate space $B^*$ is said to be weak* uniformly rotund if whenever $\{x^*_n\}$ and $\{y^*_n\}$ are sequences in $B^*$ such that $\|x^*_n\| \to 1$, $\|y^*_n\| \to 1$, and $\|x^*_n + y^*_n\| \to 2$, then $x^*_n - y^*_n \to 0$ weak*.

The following result is due to Lindenstrauss and Rosenthal [7] and Dean [2].

**The Principle of Local Reflexivity.** Let $Z$ in $B^{**}$ and $F$ in $B^*$ be finite dimensional subspaces, and let $0 < \epsilon < 1$ be arbitrary. Then there exists a linear map $T: Z \to B$ such that

(a) $T(z) = z$ for all $z \in Z \cap B$,
(b) $f(T(z)) = z(f)$ for all $z \in Z$ and $f \in F$,
(c) $(1 - \epsilon)\|z\| \leq \|T(z)\| \leq (1 + \epsilon)\|z\|$ for all $z \in Z$.

We follow the lead of Sullivan [10] in using the Principle of Local Reflexivity to obtain geometric results.

**Theorem.** The James space $J$ admits an equivalent norm such that its third conjugate space is rotund.

**Proof.** The crucial properties of $J$ that we need are that $J$ is separable and quasi-reflexive of order one. Since $J^*$ is separable, $J$ admits an equivalent norm such that its conjugate is both weak* uniformly rotund and locally uniformly rotund (see [11, p. 200]). Let $(B, \|\cdot\|)$ denote $J$ under such a norm. Write $B^{***} = B^* \oplus [b^{***}]$ where $[b^{***}]$ denotes the linear span of $\{b^{***}\}$ and $b^{***} \in B^*$.

We claim that $B^{***}$ is rotund. To show this, suppose $x^{***}$ and $y^{***}$ are norm-1 elements in $B^{***}$ such that $\|x^{***} + y^{***}\| = 2$. Write $x^{***} = x^* + \alpha b^{***}$ and $y^{***} = y^* + \beta b^{***}$ where $x^*$ and $y^*$ are in $B^*$ and $\alpha$ and $\beta$ are scalars. If $x^* \neq y^*$, then there exists $x \in S(B)$ such that $(x^* - y^*)(x) \neq 0$. By the Principle of Local Reflexivity, there exists a sequence of linear maps $T_n: \text{span}\{x^{***}, y^{***}\} \to B^*$ such that

$$Q_0(x)(T_n(x^{***} - y^{***})) = (x^{***} - y^{***})(x) = (x^* - y^*)(x)$$

and

$$(1 - \epsilon_n)\|z\| \leq \|T_n(z)\| \leq (1 + \epsilon_n)\|z\|$$

for all $z \in \text{span}\{x^{***}, y^{***}\}$, where $\{\epsilon_n\}$ is a positive sequence decreasing to zero. Let $x^*_n = T_n(x^{***})$ and $y^*_n = T_n(y^{***})$. Then we have $\|x^*_n\| \to 1$, $\|y^*_n\| \to 1$, and $\|x^*_n + y^*_n\| \to 2$. But $(x^*_n - y^*_n)(x) = (x^* - y^*)(x) \neq 0$, contradicting the fact that $B^*$ is weak* uniformly rotund. Therefore $x^* = y^*$. Choose $x^{(4)} \in S(B^{(4)})$ such that $x^{(4)}(x^{***}) = x^{(4)}(y^{***}) = 1$. Then we have

$$0 = x^{(4)}(x^{***} - y^{***}) = (\alpha - \beta)x^{(4)}(b^{***}).$$
If \( x^{(4)}(b^{***}) \neq 0 \), then \( \alpha = \beta \), and hence \( x^{***} = y^{***} \). If \( x^{(4)}(b^{***}) = 0 \), then

\[
x^{(4)}(x^*) = x^{(4)}(x^* + \alpha b^{***}) = x^{(4)}(x^{***}) = 1.
\]

From this, and since \( \|x^*\| < \|x^* + \alpha b^{***}\| = 1 \), it follows that \( \|x^*\| = 1 \) and \( \|x^* + x^{***}\| = 2 \). If \( x^{***} \neq x^* \), then by the Principle of Local Reflexivity, as in Lemma 7 of [10], there exists a sequence of linear maps \( T_n: \text{span}\{Q_1(x^*), x^{***}\} \to B^* \) such that \( T_n(Q_1(x^*)) = x^* \) for each \( n \) and

\[
(1 - \epsilon_n)\|z\| \leq \|T_n(z)\| \leq (1 + \epsilon_n)\|z\|
\]

for all \( z \in \text{span}\{Q_1(x^*), x^{***}\} \), where \( \{\epsilon_n\} \) is a positive sequence decreasing to zero. Let \( x^*_n = T_n(x^{***}) \). Then we have \( \|x^*_n\| = 1 \), \( \|x^*_n\| \to 1 \), and \( \|x^* + x^*_n\| \to 2 \). But \( \|x^* - x^*_n\| > (1 - \epsilon_n)\|x^* - x^{***}\| \), contradicting the fact that \( B^* \) is locally uniformly rotund. Therefore \( x^{***} = x^* \). Similarly, it follows that \( y^{***} = x^* \), and hence \( x^{***} = y^{***} \). This completes the proof.

**ADDED NOTE.** After this paper was submitted, Professor Daniel Amir pointed out that the theorem can also be proved using the fact that the unit ball of \( B^* \) is weak* sequentially dense in the unit ball of \( B^{***} \), since \( B^{**} \) is separable, and without using the Principle of Local Reflexivity.

**REFERENCES**


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