APPROXIMATION BY INVERTIBLES

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Abstract. The uniform closure of the set of invertibles of certain $C^*$-algebras is characterized. It is shown that indices on $\mathbb{B}(C(A))$, where $A$ is a closed annulus in the plane and on $M_2(S^1 \times S^1)$ have continuous extensions to elements which are not invertible.

1. Introduction. It is well known that the set of invertible elements $\mathbb{G}$ in a Banach algebra is open in the algebra. Feldman and Kadison have studied the uniform closure of $\mathbb{G}$ where the algebra in question is $\mathbb{B}(\mathcal{H})$ [6]. Concerning $C^*$-algebras, in general, less is known, and we discuss this problem for certain function algebras.

A related problem is one which can be phrased very generally. Let $\mathbb{A}$ be a $C^*$-algebra with identity, and let $\mathbb{G}$ denote the set of invertibles in $\mathbb{A}$. A continuous homomorphism $i$ of $\mathbb{G}$ into some discrete group $\mathbb{D}$ is called an index on $\mathbb{G}$. If $r \in \mathbb{D}$, we write $\mathbb{G}_r = i^{-1}(r)$. For any given index we can define the set

$$\mathbb{K} = \{ x \in \mathbb{G} : \text{there exists an } \epsilon > 0 \text{ and a fixed } r(x) \in \mathbb{D} \text{ : } N_{\epsilon}(x) \cap \mathbb{G} \subset \mathbb{G}_r(x) \},$$

and for $r(x) \in \mathbb{D}$ we write $\mathbb{K}_r = \{ x \in \mathbb{K} : r(x) = r \}$.

We consider the problem of determining $\mathbb{K}$ for certain $C^*$-algebras whose elements are symbols of operators on a Hilbert space. Coburn and Lebow have considered the case of the usual index defined on the invertibles of the Calkin algebra $\mathbb{B}(\mathcal{H})/\mathcal{K}$ where $\mathcal{H}$ is a separable Hilbert space and $\mathbb{K}$ is the set of compact operators on $\mathcal{H}$. Here $\mathbb{K} = \mathbb{G} [3]$.

The following is an immediate result of the definition.

Proposition 1. (i) If $i'$ is defined on $\mathbb{K}$ by $i'(x) = r(x)$, then $i' : \mathbb{K} \to \mathbb{D}$ is continuous, and $i'$ restricted to $\mathbb{G}$ equals $i$. Each $\mathbb{K}_r$ is open in $\mathbb{K}$.

(ii) $\mathbb{K}$ is not necessarily closed under products, but if $x \in \mathbb{K}_r$, $y \in \mathbb{K}_s$ and $xy \in \mathbb{K}$, then $xy \in \mathbb{K}_{r+s}$.

(iii) $\mathbb{G}_r \mathbb{G}_s \subset \mathbb{G}_{r+s}$.

(iv) $\mathbb{G}_r (\mathbb{G}_s - \mathbb{G}_s) \subset \mathbb{G}_{r+s} - \mathbb{G}_{r+s}$.

2. The algebra $\mathbb{A} = C(T)$. Let $\mathbb{A} = C(T)$ be the algebra of all complex-valued continuous functions on the unit circle $T$. A function $f \in \mathbb{A}$ is invertible if and only if $f$ does not vanish on $T$. The winding number of the
Theorem 2. Let \( i(f) = w.n.(f) \) be an index on \( \mathcal{G}(\mathbb{C}) \), and let \( \mathcal{G}_n(\mathbb{C}) = \{ f \in \mathcal{G}(\mathbb{C}); i(f) = n \} \). Then for any \( f \in \mathbb{C} - \mathcal{G}(\mathbb{C}), f \in \cap_n \mathcal{G}_n(\mathbb{C}) \), so that \( \mathcal{H} = \mathcal{G} \).

Proof. Let \( \varepsilon > 0 \) be given, and set \( S_\varepsilon = \{ z \in T; |f(z)| < \varepsilon \} \). Consider the function \( \phi \in C(T) \) defined as follows. For \( z \in T - S_\varepsilon \), \( \phi(z) = f(z) \); for \( z = e^{i\omega} \in I_1 \), where \( I_1 = \{ e^{i\omega}; \omega_0 < \omega < \omega_1 \} \) is an arbitrary interval of \( S_\varepsilon \) for which \( f(e^{i\omega_0}) = e^{i\omega_0} \) and \( f(e^{i\theta}) = e^{i\theta} \), let

\[
\phi(e^{i\omega}) = \varepsilon \exp \left\{ i \left( \frac{\omega - \omega_0}{\theta_1 - \theta_0} \right) \left( \omega_0 - \theta_0 \right) \right\}.
\]

The function \( \phi \in \mathcal{G}(\mathbb{C}) \), and \( ||f - \phi|| < 2\varepsilon \) on \( T \). Since \( \varepsilon > 0 \) is arbitrary, \( f \in \mathcal{G}(\mathbb{C}) \).

The function \( \phi \) can have arbitrary index. For if \( i(\phi) = n \) and \( k \) is any integer other than \( n \), the function \( \phi'(z) = \phi(z) \) for \( z \in T - I_1 \), where \( I_1 = \{ e^{i\omega}; \omega_0 < \omega < \omega_1 \} \) is some fixed interval of \( S_\varepsilon \), and by

\[
\phi'(z) = \begin{cases}
\varepsilon \exp \left\{ zi \left[ \frac{\theta_1 - \theta_0}{\omega_1 - \omega_0} \right] (\omega - \omega_0) + i\theta_0 \right\}, & \omega_0 < \omega < \frac{\omega_0 + \omega_1}{2}, \\
\varepsilon \exp i\omega_1 \exp \left\{ \frac{-4\pi i(k - n)}{\omega_1 - \omega_0} \left( \omega - \frac{\omega_0 + \omega_1}{2} \right) \right\}, & \frac{\omega_0 + \omega_1}{2} < \omega < \omega_1,
\end{cases}
\]

has index \( k \), and \( ||f - \phi'|| < 2\varepsilon \) on \( T \). Since \( \varepsilon > 0 \) and \( k \) are arbitrary, we conclude that \( f \in \cap_n \mathcal{G}_n(\mathbb{C}) \).

Since every neighborhood of \( f \) contains invertibles of all indices, \( \mathcal{H} = \mathcal{G} \).

We next recall some basic notions of Fredholm theory. An operator \( A \) in \( \mathcal{B}(\mathcal{K}) \) is said to be Fredholm if the range of \( A \) is closed and \( \dim \ker A + \dim \ker A^* \) is finite. Atkinson has shown that these conditions are equivalent to \( \pi(A) \) being invertible in the algebra \( \mathcal{B}(\mathcal{K})/\mathcal{K} \) where \( \pi \) is the quotient map \([1]\). The analytic index \( i_{an} \) defined on the set of Fredholm operators \( \mathcal{G} \) by \( i_{an}(A) = \dim \ker A - \dim \ker A^* \) is a homomorphism of \( \mathcal{G} \) onto the integers.

Now for \( \phi \in C(T) \), the Toeplitz operator \( T_\phi \) is defined as follows: Let \( L^2(T) \) be the space of Lebesgue square-integrable functions on \( T \) with the orthonormal basis \( \{ e^{inx}, n \in \mathbb{Z} \} \). The subspace spanned by those \( e^{inx} \) for which \( n \geq 0 \) is denoted by \( H^2(T) \). The operator \( T_\phi \) is defined on \( H^2(T) \) by \( T_\phi f = P(\phi f) \) where \( P \) is the orthogonal projection from \( L^2 \) onto \( H^2 \).

The Banach algebra \( \mathcal{B} \) generated by \( T \), has been studied by Coburn \([2]\). He has shown that \( \mathcal{B} = \{ T_\phi + K; \phi \in C(T), K \text{ compact} \} \) and that the \( C^* \)-algebra \( \mathcal{B}/\mathcal{K} \) is isometrically isomorphic to \( C(T) \). It follows that \( T_\phi \) is
a Fredholm operator if and only if \( \phi \) does not vanish on \( T \) and that \( i_{an}(T_\phi) \)
equals the negative of \( w.n.(\phi) \) with respect to the origin. Lastly, let the map \( \rho: B \to C(T) \) be given by \( \rho(T_\phi + K) = \phi \). It has been shown that the natural cross section \( s \) of \( \rho \) is continuous. Thus for any component \( C \) of \( \mathcal{S}(C(T)) \),
\[
s(C) = s(C)^{-1} \]
These remarks together with Theorem 2 yield the following analogue of Theorem 2 for the set of Fredholm operators \( \mathcal{F} \) in \( B \).

**Corollary 3.** There is no possibility of continuously extending the analytic index beyond \( \mathcal{F} \).

Let \( \Omega \) be a compact set in the plane whose boundary consists of finitely many Jordan curves. If \( i \) is the index defined on \( \mathcal{S}(C(\Omega)) \) by \( i(f) = 0 \), then \( \mathcal{H} = \mathcal{F} \) is always larger than \( \mathcal{S} \). The problem is to determine \( \mathcal{S} \). We limit ourselves to the following

**Theorem 4.** If \( \Omega \) is a set of the type described, then \( \mathcal{S} \subseteq \mathcal{H} \subseteq C(\Omega) \).

**Proof.** Let \( a \) be an interior point of \( \Omega \). The function \( f(z) = z - a \) in \( C(\Omega) \) has nonzero winding number along closed curves in \( \Omega \) about the point \( a \). Hence \( f \) cannot be uniformly approximated by nonvanishing functions of \( C(\Omega) \).

On the other hand, if \( f \in C(\Omega) \) vanishes only on \( \partial \Omega \), then \( f \in \mathcal{S} \). For any component \( \mathcal{C} \) of \( \Omega \) with interior and any \( \varepsilon > 0 \), let \( I = \{ z \in \partial \mathcal{C} : |f(z)| < \varepsilon/2 \} \). Then \( I = \bigcup I_i \) is a countable union of disjoint smooth curves. There exist smooth arcs \( \gamma_i \) in \( \mathcal{C} \) having as endpoints the endpoints of \( I_i \) such that on the region \( R_i \) bounded by \( I_i \) and \( \gamma_i \), \( |f(z)| < \varepsilon \). On each \( R_i \) there is a continuous function \( f_i \) satisfying \( f_{|\gamma_i} = f_i_{|\gamma_i} \) with the range of \( f_i \) on all of \( R_i \) equal to the range of \( f \) on \( \gamma_i \). The function \( f' \) defined by
\[
\begin{cases}
f(z), & z \in \mathcal{C} - \bigcup R_i, \\f_i(z), & z \in R_i,
\end{cases}
\]
is continuous and nonvanishing on \( \mathcal{C} \), and \( \|f - f'\| < 2\varepsilon \) on \( \mathcal{C} \). Since \( \mathcal{C} \) and \( \varepsilon > 0 \) are arbitrary, \( f \in \mathcal{S} \).

**3. The algebra \( \mathcal{F} = C(A) \).** We now describe an algebra for which there exists a nontrivial index that can be continuously extended beyond \( \mathcal{S}(\mathcal{F}) \). Let \( A \) be the closed annulus in the plane centered at the origin with radii \( r = \frac{1}{2} \) and \( r = 1 \). A function \( f \in \mathcal{F} = C(A) \) is invertible if and only if \( f \) does not vanish on \( A \), and the winding number traced out by \( f \) with respect to the origin along the curve \( r = \frac{3}{4} \) determines an index on \( \mathcal{S}(\mathcal{F}) \).

**Theorem 5.** For the index described, \( \mathcal{S} \preceq \mathcal{H} \preceq \mathcal{F} \preceq C(A) \).

**Proof.** If \( f \in C(A) \) vanishes only on \( \partial A \), then by Theorem 4, \( f \in \mathcal{S}(\mathcal{F}) \). The winding number of \( f \) along \( r = \frac{3}{4} \) is some integer, say \( n \). It follows from the continuity of the winding number that if \( g \in \mathcal{S}(\mathcal{F}) \) satisfies \( \|f - g\| < \varepsilon \) for \( \varepsilon > 0 \) sufficiently small, then \( w.n.(g) \) along \( r = \frac{3}{4} \) is also \( n \). Hence, \( f \in (\mathcal{H}_n \preceq \mathcal{S}_n) \).
On the other hand, let \( g \in C(A) \) be defined by \( g(re^{i\theta}) = g(e^{i\theta}), \frac{1}{2} < r < 1; \) \( g(re^{i\theta'}) = 0 \) for some fixed \( \theta' \), and \( g(re^{i\theta}) \neq 0 \) for \( \theta \neq \theta' \). It follows from Theorem 2 that \( g \in \cap_n \mathcal{S}_n(\mathcal{H}) \), so that \( g \not\in \mathcal{H} \). Hence, \( \mathcal{H} \subsetneq \mathcal{H} \). Finally, it follows from Theorem 4 that if \( h \in C(A) \) has an isolated zero at an interior point of \( A \), then \( h \not\in \mathcal{H} \).

The elements of \( C(A) \) can be realized as symbols of operators on a Hilbert space. Because of this, we can characterize a distinguished class of non-Fredholm operators to which the analytic index can be continuously extended.

Let \( \mathcal{H} = L^2(A, dA) \oplus H^2(\frac{1}{4} \pi, d\theta) \), where \( L^2(A, dA) \) is the Hilbert space of all square-integrable functions on \( A \) with respect to area measure and \( H^2(\frac{1}{4} \pi, d\theta) \) is the Hardy space of all square-integrable functions on \( \frac{1}{4} \pi \) with respect to normalized Haar measure. For \( \phi \in C(A) \), \( M_\phi \) denotes the multiplication operator on \( L^2(A, dA) \) induced by \( \phi \) and \( T_\phi \) denotes the Toeplitz operator on \( H^2(\frac{1}{4} \pi, d\theta) \) induced by \( \phi = \phi|_{\frac{1}{4} \pi} \). We shall denote by \( S_\phi \) the operator \( M_\phi \oplus T_\phi \) on \( \mathcal{H} \).

**Lemma 6.** For any \( \phi, \psi \in C(A) \) the following properties hold:

(i) \( S_\phi^* = S_\psi \);

(ii) \( S_\lambda \phi \oplus \psi = \lambda S_\phi + \mu S_\psi \) for any complex scalars \( \lambda \) and \( \mu \);

(iii) \( \|S_\phi\| = \|\pi(S_\phi)\| = \|\phi\| \).

**Proof.** (i) and (ii) are direct computations; (iii) is a consequence of the Douglas generalization of the Berg-Weyl theorem [5].

**Theorem 7.** The set \( \mathcal{A} = \{S_\phi + K, K \in \mathcal{S}(\mathcal{H})\} \) is a Banach algebra, and the representation of elements of \( \mathcal{A} \) as sums is unique.

**Proof.** It follows from the lemma that if the operators \( A \) and \( B \) are in \( \mathcal{A} \), then so are \( A + B, \lambda \cdot A, \) and \( A^* \). Since \( T_\phi T_\psi - T_{\phi \psi} \) is a compact operator on \( H^2(\frac{1}{4} \pi, d\theta) \) [2], \( S_\phi S_\psi - S_{\phi \psi} \) is in \( \mathcal{S}(\mathcal{H}) \), and \( \mathcal{A} \) is closed under multiplication. It remains to be shown that the algebra \( \mathcal{A} \) is closed. By (iii) above, if a sequence \( \{S_{\phi_n} + K_n\} \) is Cauchy, then so is \( \{\phi_n\} \). But then \( \phi_n \to \phi \) for some \( \phi \in C(A) \). Again by (iii), \( S_\phi \to S_\phi \) so that \( K_n \to K \) for some \( K \in \mathcal{S}(\mathcal{H}) \), and \( \mathcal{A} \) is closed. Lastly, the norm equality \( \|S_\phi + K\| = \|\phi\| \) implies that the representation is unique.

**Corollary 8.** The map \( S_\phi + K \leftrightarrow \phi \) is an isometric \(*\)-isomorphism between \( \mathcal{A} / \mathcal{H} \) and \( C(A) \). The operator \( S_\phi \) is Fredholm in \( \mathcal{A} \) if and only if \( \phi \) does not vanish on \( A \), and the analytic index on \( S_\phi \) equals minus the winding number of \( \phi \) along \( r = \frac{1}{4} \) with respect to the origin.

**Proof.** The first remarks follow immediately from Proposition 6, Theorem 7, and Atkinson's characterization of Fredholm operators. Since \( \text{ind} S_\phi = \text{ind} M_\phi + \text{ind} T_\phi = 0 + \text{w.n.}(\phi) \) [5], we have the index characterization of the corollary.

**Corollary 9.** The analytic index on the set of Fredhols \( \mathcal{F} \subset \mathcal{H} \) can be continuously extended to those elements of \( \mathcal{A} / \mathcal{H} \) whose symbols lies in \( \mathcal{H} \).

**Proof.** This is an immediate consequence of Theorem 5 and the fact that the map \( \phi \to S_\phi \) is continuous.
4. The algebra $\mathcal{B} = M_2^{S^2 \times S^1}$. We will consider $S^2$ as the one point compactification of the plane and $S^1$ the unit sphere $|z| = 1$. We shall study the $C^*$-algebra $\mathcal{B} = M_2^{S^2 \times S^1}$ of $2 \times 2$ matrix-valued functions on $S^2 \times S^1$ in the matrix supremum norm. This algebra can be identified as the symbol space of the algebra $\mathcal{B}$ of $2 \times 2$ systems of singular integral operators. We briefly describe this latter algebra and refer the reader to [4] for details.

Let $\mathcal{B}$ denote the Fourier transform of the function $f(x) \in L^2(R^2)$. The algebra $\mathcal{B}$ is generated by $2 \times 2$ systems of (a) the multiplication operators $M_\phi$, $\phi \in C(S^2)$, (b) $T^{-1}M_k(x)T$ where $k(x) \in C(S^1)$ is homogeneous of degree zero, and (c) the ideal $\mathcal{K}$ of compact operators on $L^2(R^2) \oplus L^2(R^2)$. The symbol map $\sigma$ on $\mathcal{B}$ given by

\[
\sigma(M_\phi) = \phi(x), \quad x \in S^2;
\]
\[
\sigma(T^{-1}M_k T) = k(\theta), \quad \theta \in S^1;
\]
\[
\sigma(K) = 0, \quad K \text{ compact};
\]

can be extended to a $*$-homomorphism from all of $\mathcal{B}$ onto $M_2^{S^2 \times S^1}$ and has a continuous cross section. An operator in $\mathcal{B}$ is Fredholm if and only if its symbol has nonvanishing determinant on $S^2 \times S^1$, and two Fredholm operators are homotopic if and only if their symbols, considered as matrix functions from $S^2 \times S^1$ into $GL_2$, the general linear group of all $2 \times 2$ nonsingular matrices, are homotopic.

An index $i: \mathcal{B} \to GL_2^{S^2 \times S^1} \to Z$ is determined as follows. If for $f \in \mathcal{B}$, we write $f = pu$, where $p$ is positive and

\[
u \in \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \in U_2^{S^2 \times S^1},
\]

the $2 \times 2$ unitary matrices in $\mathcal{B}$, then the map $\gamma(u) = (u_1, u_2)$ determines an element of $\pi(S^{3g \times S^1})$. Now $\pi_0(S^{3g \times S^1}) \cong Z$, and we define $i(f) = [\gamma \circ u] \in \pi_0(S^{3g \times S^1})$, where $[\gamma \circ u]$ denotes the class of $\gamma \circ u$ in $\pi_0(S^{3g \times S^1})$ [9].

We note that it is immaterial whether the “top” or “bottom” row map is used to define the index. For

\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix},
\]

and $U = (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ is unitary. If we write $U = e^{iH}$ for some Hermitian $H$, then $U_t = e^{itH}$ is a homotopy between $U$ and the identity.

**Theorem 10.** For the algebra $\mathcal{B} = M_2^{S^2 \times S^1}$, where $\mathcal{B} = GL_2^{S^2 \times S^1}$ and for the index $i$ described above, $\mathcal{K} \neq \mathcal{B}$.

**Proof.** We shall describe a class of elements contained in $\mathcal{K} - \mathcal{B}$. Consider those elements in $M_2^{S^2 \times S^1}$ of the form

\[
h(x) = \begin{pmatrix} \alpha(x) & \beta(x) \\ \beta(x) & \tau(x) \end{pmatrix},
\]
where \( \alpha(x) > m > 0, \tau(x) > |\beta(x)| \) and \( m > 3\|\tau\|_\infty \). If \( \tau(x) = 0 \) on \( S^2 \times S^1 \), then \( h \notin \mathcal{F} \). On the other hand, \( h(x) > 0 \) so that \( h \in \mathcal{F} \). Now if \( \|u - h\| < \varepsilon \) for
\[
p = \begin{pmatrix} p_1 & \bar{p}_2 \\ p_2 & p_3 \end{pmatrix}
\]
positive
and
\[
u^* = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}
\]
unitary,
then
\[
|p_3| < |p_3 - (\bar{\beta}u_4 + \tau u_3)| + |\beta| + |\tau| < \varepsilon + \|\tau\|_\infty.
\]
Hence,
\[
|\alpha u_2| < |(\bar{\beta}_3 - \beta u_4) - \alpha u_2| + |p_3 - \bar{\beta}u_4| < 2\varepsilon + \|\tau\|_\infty.
\]
It follows that
\[
\|u_2\|_\infty < \|\alpha^{-1}\|_\infty \varepsilon + \|\alpha^{-1}\|_\infty \|\tau\|_\infty < (6\varepsilon + m)/3m.
\]
If \( \varepsilon > 0 \) is chosen so that \( \varepsilon < m/3 \), then \( \|u\|_\infty < 1 \). This implies that the range of \( \gamma \circ u^* \) is not all of \( S^3 \). Therefore, \( i(u^*) = -i(u) = [\gamma \circ u^*] = 0 \), since \( \gamma \) is a homeomorphism. Hence if \( \varepsilon < m/3 \), \( h \) is an element of \( \mathcal{K}_0 - \varepsilon_0 \).

It follows from Proposition 1 that \( \mathcal{K}_r - \mathcal{F}_r \) is nonempty for all \( r \).

**Corollary 11.** There exists a continuous extension of the analytic index on the set of Fredholm operators \( \mathcal{F} \) in \( \mathcal{G} \) to a class of operators in \( \mathcal{F} - \mathcal{F} \).

**Proof.** The analytic index \( i_{an} \) on \( \mathcal{G} \) is given by \( \sigma \circ i \), where \( \sigma \) is the symbol map and \( i \) is the topological index on \( GL_2(S^2 \times S^1) \). Since \( \sigma \) has a continuous cross section, \( i_{an} \) can be continuously extended to those operators in \( \mathcal{G} - \mathcal{G} \) whose symbols lie in \( \mathcal{K} \).

**References**


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