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APPROXIMATION BY INVERTIBLES

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ABSTRACT. The uniform closure of the set of invertibles of certain C^*-algebras is characterized. It is shown that indices on $\mathcal{G}(C(A))$, where A is a closed annulus in the plane and on $M_2^{S^2\times S^1}$ have continuous extensions to elements which are not invertible.

1. **Introduction.** It is well known that the set of invertible elements $\mathcal{G}$ in a Banach algebra is open in the algebra. Feldman and Kadison have studied the uniform closure of $\mathcal{G}$ where the algebra in question is $\mathcal{B}(\mathcal{H})$ [6]. Concerning C^*-algebras, in general, less is known, and we discuss this problem for certain function algebras.

A related problem is one which can be phrased very generally. Let $\mathcal{A}$ be a C^*-algebra with identity, and let $\mathcal{G}$ denote the set of invertibles in $\mathcal{A}$. A continuous homomorphism i of $\mathcal{G}$ into some discrete group $\mathcal{D}$ is called an index on $\mathcal{G}$. If $r \in \mathcal{D}$, we write $\mathcal{G}_r = i^{-1}(r)$. For any given index we can define the set

$$\mathcal{H} = \{x \in \bar{\mathcal{G}} : \text{there exists an } \varepsilon > 0 \text{ and a fixed } r(x) \in \mathcal{D} : N_{\varepsilon(x)}(x) \cap \mathcal{G} \subset \mathcal{G}_{r(x)}\},$$

and for $r(x) \in \mathcal{D}$ we write $\mathcal{H}_r = \{x \in \mathcal{H} : r(x) = r\}$.

We consider the problem of determining $\mathcal{H}$ for certain C^*-algebras whose elements are symbols of operators on a Hilbert space. Coburn and Lebow have considered the case of the usual index defined on the invertibles of the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$ where $\mathcal{H}$ is a separable Hilbert space and $\mathcal{K}$ is the set of compact operators on $\mathcal{H}$. Here $\mathcal{H} = \mathcal{G}$ [3].

The following is an immediate result of the definition.

PROPOSITION 1. (i) *If i' is defined on $\mathcal{H}$ by $i'(x) = r(x)$, then i': $\mathcal{H} \to \mathcal{D}$ is continuous, and i' restricted to $\mathcal{G}$ equals i. Each $\mathcal{H}_r$ is open in $\mathcal{H}$.*

(ii) *$\mathcal{H}$ is not necessarily closed under products, but if $x \in \mathcal{H}_r$, $y \in \mathcal{H}_s$ and $xy \in \mathcal{H}$, then $xy \in \mathcal{H}_{r+s}$.*

(iii) $\mathcal{G}_r \mathcal{H}_s \subset \mathcal{H}_{r+s}$.

(iv) $\mathcal{G}_r(\mathcal{H}_s - \mathcal{G}_s) \subset \mathcal{H}_{r+s} - \mathcal{G}_{r+s}$.

2. **The algebra** $\mathcal{A} = C(T)$. Let $\mathcal{A} = C(T)$ be the algebra of all complex-valued continuous functions on the unit circle T. A function $f \in \mathcal{A}$ is invertible if and only if f does not vanish on T. The winding number of the

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curve traced out by $f \in \mathcal{G}(\mathcal{A})$ with respect to origin, denoted by w.n.(f) determines an index on $\mathcal{G}$ [2].

THEOREM 2. *Let* $i(f) =$ w.n.(f) *be an index on* $\mathcal{G}(\mathcal{A})$, *and let* $\mathcal{G}_n(\mathcal{A}) = \{f \in \mathcal{G}(\mathcal{A})\colon i(f) = n\}$. *Then for any* $f \in \mathcal{A} - \mathcal{G}(\mathcal{A})$, $f \in \bigcap_n \overline{\mathcal{G}_n(\mathcal{A})}$, *so that* $\mathcal{H} = \mathcal{G}$.

PROOF. Let $\varepsilon > 0$ be given, and set $S_\varepsilon = \{z \in T\colon |f(z)| < \varepsilon\}$. Consider the function $\phi \in C(T)$ defined as follows. For $z \in T - S_\varepsilon$, $\phi(z) = f(z)$; for $z = e^{i\omega} \in I_i$ where $I_i = \{e^{i\omega}\colon \omega_0 < \omega < \omega_1\}$ is an arbitrary interval of S_ε for which $f(e^{i\omega_0}) = \varepsilon e^{i\omega_0}$ and $f(e^{i\theta_1}) = \varepsilon e^{i\theta_1}$, let

$$\phi(e^{i\omega}) = \varepsilon \exp\left\{ i\left[\left(\frac{\omega_1 - \omega_0}{\theta_1 - \theta_0}\right)(\omega - \omega_0) + \theta_0\right]\right\}.$$

The function $\phi \in \mathcal{G}(\mathcal{A})$, and $\|f - \phi\| \leqslant 2\varepsilon$ on T. Since $\varepsilon > 0$ is arbitrary, $f \in \overline{\mathcal{G}(\mathcal{A})}$.

The function ϕ can have arbitrary index. For if $i(\phi) = n$ and k is any integer other than n, the function $\phi' \in \mathcal{G}(\mathcal{A})$ defined by $\phi'(z) = \phi(z)$ for $z \in T - I_1$ where $I_1 = \{e^{i\omega}\colon \omega_0 < \omega < \omega_1\}$ is some fixed interval of S_ε, and by

$$\phi'(z) = \begin{cases} \varepsilon \exp\left\{ zi\left[\dfrac{\theta_1 - \theta_0}{\omega_1 - \omega_0}\right](\omega - \omega_0) + i\theta_0\right\}, \\ \qquad\qquad\qquad\qquad \omega_0 < \omega < \dfrac{\omega_0 + \omega_1}{2}, \\ \varepsilon \exp i\omega_1 \exp\left\{\dfrac{-4\pi i(k-n)}{\omega_1 - \omega_0}\left(\omega - \left(\dfrac{\omega_0 - \omega_1}{2}\right)\right)\right\}, \\ \qquad\qquad\qquad\qquad \dfrac{\omega_0 + \omega_1}{2} < \omega < \omega_1, \end{cases}$$

has index k, and $\|f - \phi'\| < 2\varepsilon$ on T. Since $\varepsilon > 0$ and k are arbitrary, we conclude that $f \in \bigcap_n \overline{\mathcal{G}_n(\mathcal{A})}$.

Since every neighborhood of f contains invertibles of all indices, $\mathcal{H} = \mathcal{G}$.

We next recall some basic notions of Fredholm theory. An operator A in $\mathfrak{B}(\mathcal{H})$ is said to be Fredholm if the range of A is closed and dim ker A + dim ker A^* is finite. Atkinson has shown that these conditions are equivalent to $\pi(A)$ being invertible in the algebra $\mathfrak{B}(\mathcal{H})/\mathcal{K}$ where π is the quotient map [1]. The analytic index i_{an} defined on the set of Fredholm operators $\mathfrak{F}$ by $i_{\text{an}}(A) = \dim \ker A - \dim \ker A^*$ is a homomorphism of $\mathfrak{F}$ onto the integers.

Now for $\phi \in C(T)$, the Toeplitz operator T_ϕ is defined as follows: Let $L^2(T)$ be the space of Lebesgue square-integrable functions on T with the orthonormal basis $\{e^{inx}, n \in Z\}$. The subspace spanned by those e^{inx} for which $n \geqslant 0$ is denoted by $H^2(T)$. The operator T_ϕ is defined on $H^2(T)$ by $T_\phi f = P(\phi f)$ where P is the orthogonal projection from L^2 onto H^2.

The Banach algebra $\mathfrak{B}$ generated by T_z has been studied by Coburn [2]. He has shown that $\mathfrak{B} = \{T_\phi + K\colon \phi \in C(T), K \text{ compact}\}$ and that the C^*-algebra $\mathfrak{B}/\mathcal{K}$ is isometrically *-isomorphic to $\mathcal{C}(T)$. It follows that T_ϕ is a Fredholm operator if and only if ϕ does not vanish on T and that $i_{\mathrm{an}}(T_\phi)$ equals the negative of w.n.(ϕ) with respect to the origin. Lastly, let the map ρ: $\mathfrak{B} \to C(T)$ be given by $\rho(T_\phi + K) = \phi$. It has been shown that the natural cross section s of ρ is continuous. Thus for any component $\mathcal{C}$ of $\mathcal{G}(C(T))$, $s(\overline{\mathcal{C}}) = \overline{s(C)}$ [7].

These remarks together with Theorem 2 yield the following analogue of Theorem 2 for the set of Fredholm operators $\mathcal{F}$ in $\mathfrak{B}$.

COROLLARY 3. *There is no possibility of continuously extending the analytic index beyond $\mathcal{F}$.*

Let Ω be a compact set in the plane with interior whose boundary consists of finitely many Jordan curves. If i is the index defined on $\mathcal{G}(C(\Omega))$ by $i(f) = 0$, then $\mathcal{H} = \overline{\mathcal{G}}$ is always larger than $\mathcal{G}$. The problem is to determine $\mathcal{G}$. We limit ourselves to the following

THEOREM 4. *If Ω is a set of the type described, then $\mathcal{G} \subsetneq \overline{\mathcal{G}} \subsetneq C(\Omega)$.*

PROOF. Let a be an interior point of Ω. The function $f(z) = z - a$ in $C(\Omega)$ has nonzero winding number along closed curves in Ω about the point a. Hence f cannot be uniformly approximated by nonvanishing functions of $C(\Omega)$.

On the other hand, if $f \in C(\Omega)$ vanishes only on $\partial\Omega$, then $f \in \overline{\mathcal{G}}$. For any component $\mathcal{C}$ of Ω with interior and any $\varepsilon > 0$, let $I = \{z \in \partial\mathcal{C}: |f(z)| < \varepsilon/2\}$. Then $I = \bigcup_i I_i$ is a countable union of disjoint smooth curves. There exist smooth arcs γ_i in $\mathcal{C}$ having as endpoints the endpoints of $\overline{I}_i$ such that on the region R_i bounded by $\overline{I}_i$ and γ_i, $|f(z)| < \varepsilon$. On each R_i there is a continuous function f_i satisfying $f_{i|\gamma_i} = f_{|\gamma_i}$ with the range of f_i on all of R_i equal to the range of f on γ_i. The function f' defined by

$$f'(z) = \begin{cases} f(z), & z \in \mathcal{C} - \bigcup_i R_i, \\ f_i(z), & z \in R_i, \end{cases}$$

is continuous and nonvanishing on $\mathcal{C}$, and $\|f - f'\| < 2\varepsilon$ on $\mathcal{C}$. Since $\mathcal{C}$ and $\varepsilon > 0$ are arbitrary, $f \in \overline{\mathcal{G}}$.

3. **The algebra** $\mathcal{Q} = C(A)$. We now describe an algebra for which there exists a nontrivial index that can be continuously extended beyond $\mathcal{G}(\mathcal{Q})$. Let A be the closed annulus in the plane centered at the origin with radii $r = \frac{1}{2}$ and $r = 1$. A function $f \in \mathcal{Q} = C(A)$ is invertible if and only if f does not vanish on A, and the winding number traced out by f with respect to the origin along the curve $r = \frac{3}{4}$ determines an index on $\mathcal{G}(\mathcal{Q})$.

THEOREM 5. *For the index described, $\mathcal{G} \subsetneq \mathcal{H} \subsetneq \overline{\mathcal{G}} \subsetneq C(A)$.*

PROOF. If $f \in C(A)$ vanishes only on ∂A, then by Theorem 4, $f \in \overline{\mathcal{G}(\mathcal{Q})}$. The winding number of f along $r = \frac{3}{4}$ is some integer, say n. It follows from the continuity of the winding number that if $g \in \mathcal{G}(\mathcal{Q})$ satisfies $\|f - g\| < \varepsilon$ for $\varepsilon > 0$ sufficiently small, then w.n.(g) along $r = \frac{3}{4}$ is also n. Hence, $f \in (\mathcal{H}_n - \mathcal{G}_n)$.

On the other hand, let $g \in C(A)$ be defined by $g(re^{i\theta}) = g(e^{i\theta})$, $\frac{1}{2} \leqslant r \leqslant 1$; $g(re^{i\theta'}) = 0$ for some fixed θ', and $g(re^{i\theta}) \neq 0$ for $\theta \neq \theta'$. It follows from Theorem 2 that $g \in \overline{\bigcap_n \mathcal{G}_n(\mathcal{A})}$, so that $g \notin \mathcal{H}$. Hence, $\mathcal{H} \subsetneq \overline{\mathcal{G}}$. Finally, it follows from Theorem 4 that if $h \in C(A)$ has an isolated zero at an interior point of A, then $h \notin \overline{\mathcal{G}}$.

The elements of $C(A)$ can be realized as symbols of operators on a Hilbert space. Because of this, we can characterize a distinguished class of non-Fredholm operators to which the analytic index can be continuously extended.

Let $\mathcal{H} = L^2(A, dA) \oplus H^2(\frac{3}{4}T, d\theta)$, where $L^2(A, dA)$ is the Hilbert space of all square-integrable functions on A with respect to area measure and $H^2(\frac{3}{4}T, d\theta)$ is the Hardy space of all square-integrable functions on $\frac{3}{4}T$ with respect to normalized Haar measure. For $\phi \in C(A)$, M_ϕ denotes the multiplication operator on $L^2(A, dA)$ induced by ϕ and $T_{\tilde{\phi}}$ denotes the Toeplitz operator on $H^2(\frac{3}{4}T, d\theta)$ induced by $\tilde{\phi} = \phi|_{\frac{3}{4}T}$. We shall denote by S_ϕ the operator $M_\phi \oplus T_{\tilde{\phi}}$ on $\mathcal{H}$.

LEMMA 6. *For any ϕ, $\psi \in C(A)$ the following properties hold*:
(i) $S_\phi^* = S_{\bar{\phi}}$;
(ii) $S_{\lambda\phi + \mu\psi} = \lambda S_\phi + \mu S_\psi$ *for any complex scalars* λ *and* μ;
(iii) $\|S_\phi\| = \|\pi(S_\phi)\| = \|\phi\|$.

PROOF. (i) and (ii) are direct computations; (iii) is a consequence of the Douglas generalization of the Berg-Weyl theorem [5].

THEOREM 7. *The set $\mathcal{A} = \{S_\phi + K, K \in \mathcal{K}(\mathcal{H})\}$ is a Banach algebra, and the representation of elements of $\mathcal{A}$ as sums is unique.*

PROOF. It follows from the lemma that if the operators A and B are in $\mathcal{A}$, then so are $A + B$, $\lambda \cdot A$, and A^*. Since $T_{\tilde{\phi}} T_{\tilde{\psi}} - T_{\tilde{\phi}\tilde{\psi}}$ is a compact operator on $H^2(\frac{3}{4}T, d\theta)$ [2], $S_\phi S_\psi - S_{\phi\psi}$ is in $\mathcal{K}(\mathcal{H})$, and $\mathcal{A}$ is closed under multiplication. It remains to be shown that the algebra $\mathcal{A}$ is closed. By (iii) above, if a sequence $\{S_{\phi_n} + K_n\}$ is Cauchy, then so is $\{\phi_n\}$. But then $\phi_n \to \phi$ for some $\phi \in C(A)$. Again by (iii), $S_{\phi_n} \to S_\phi$ so that $K_n \to K$ for some $K \in \mathcal{K}(\mathcal{H})$, and $\mathcal{A}$ is closed. Lastly, the norm equality $\|S_\phi + K\| = \|\phi\|$ implies that the representation is unique.

COROLLARY 8. *The map $S_\phi + K \leftrightarrow \phi$ is an isometric *-isomorphism between $\mathcal{A}/\mathcal{K}$ and $C(A)$. The operator S_ϕ is Fredholm in $\mathcal{A}$ if and only if ϕ does not vanish on A, and the analytic index on S_ϕ equals minus the winding number of ϕ along $r = \frac{3}{4}$ with respect to the origin.*

PROOF. The first remarks follow immediately from Proposition 6, Theorem 7, and Atkinson's characterization of Fredholm operators. Since ind S_ϕ = ind M_ϕ + ind $T_{\tilde{\phi}} = 0 +$ w.n.$(\tilde{\phi})$ [5], we have the index characterization of the corollary.

COROLLARY 9. *The analytic index on the set of Fredholms $\mathcal{F} \subset \mathcal{B}$ can be continuously extended to those elements of $\overline{\mathcal{F}} - \mathcal{F}$ whose symbols lies in $\mathcal{H}$.*

PROOF. This is an immediate consequence of Theorem 5 and the fact that the map $\phi \to S_\phi$ is continuous.

4. **The algebra** $\mathcal{A} = M_2^{S^2 \times S^1}$. We will consider S^2 as the one point compactification of the plane and S^1 the unit sphere $|z| = 1$. We shall study the C^*-algebra $\mathcal{A} = M_2^{S^2 \times S^1}$ of 2×2 matrix-valued functions on $S^2 \times S^1$ in the matrix supremum norm. This algebra can be identified as the symbol space of the algebra $\mathcal{B}$ of 2×2 systems of singular integral operators. We briefly describe this latter algebra and refer the reader to [4] for details.

Let $\mathcal{F}f$ denote the Fourier transform of the function $f(x) \in L^2(R^2)$. The algebra $\mathcal{B}$ is generated by 2×2 systems of (a) the multiplication operators M_ϕ, $\phi \in C(S^2)$, (b) $\mathcal{F}^{-1} M_{k(x)} \mathcal{F}$ where $k(x) \in C(S^1)$ is homogeneous of degree zero, and (c) the ideal $\mathcal{K}$ of compact operators on $L^2(R^2) \oplus L^2(R^2)$. The symbol map σ on $\mathcal{B}$ given by

$$\begin{aligned} \sigma(M_\phi) &= \phi(x), \qquad x \in S^2; \\ \sigma(\mathcal{F}^{-1} M_k \mathcal{F}) &= k(\theta), \qquad \theta \in S^1; \\ \sigma(K) &= 0, \qquad K \text{ compact}; \end{aligned}$$

can be extended to a *-homomorphism from all of $\mathcal{B}$ onto $M_2^{S^2 \times S^1}$ and has a continuous cross section. An operator in $\mathcal{B}$ is Fredholm if and only if its symbol has nonvanishing determinant on $S^2 \times S^1$, and two Fredholm operators are homotopic if and only if their symbols, considered as matrix functions from $S^2 \times S^1$ into GL_2, the general linear group of all 2×2 nonsingular matrices, are homotopic.

An index i: $\mathcal{G} = \mathrm{GL}_2^{S^2 \times S^1} \to Z$ is determined as follows. If for $f \in \mathcal{G}$, we write $f = pu$, where p is positive and

$$u = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \in U_2^{S^2 \times S^1},$$

the 2×2 unitary matrices in $\mathcal{G}$, then the map $\gamma(u) = (u_1, u_2)$ determines an element of $S^{3^{S^2 \times S^1}}$. Now $\pi_0(S^{3^{S^2 \times S^1}}) \simeq Z$, and we define $i(f) = [\gamma \circ u] \in \pi_0(S^{3^{S^2 \times S^1}})$, where $[\gamma \circ u]$ denotes the class of $\gamma \circ u$ in $\pi_0(S^{3^{S^2 \times S^1}})$ [9].

We note that it is immaterial whether the "top" or "bottom" row map is used to define the index. For

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix},$$

and $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is unitary. If we write $U = e^{iH}$ for some Hermitian H, then $U_t = e^{itH}$ is a homotopy between U and the identity.

THEOREM 10. *For the algebra* $\mathcal{A} = M_2^{S^2 \times S^1}$, *where* $\mathcal{G} = \mathrm{GL}_2^{S^2 \times S^1}$ *and for the index i described above*, $\mathcal{K} \neq \mathcal{G}$.

PROOF. We shall describe a class of elements contained in $\mathcal{K} - \mathcal{G}$. Consider those elements in $M_2^{S^2 \times S^1}$ of the form

$$h(x) = \begin{pmatrix} \alpha(x) & \overline{\beta(x)} \\ \beta(x) & \tau(x) \end{pmatrix},$$

where $\alpha(x) \geqslant m > 0$, $\tau(x) \geqslant |\beta(x)|$ and $m > 3\|\tau\|_\infty$. If $\tau(x) = 0$ on $S^2 \times S^1$, then $h \notin \mathcal{G}$. On the other hand, $h(x) \geqslant 0$ so that $h \in \bar{\mathcal{G}}$. Now if $\|pu - h\| < \varepsilon$ for

$$p = \begin{pmatrix} p_1 & \bar{p}_2 \\ p_2 & p_3 \end{pmatrix} \quad \text{positive}$$

and

$$u^* = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \quad \text{unitary,}$$

then

$$|p_3| < \left|p_3 - \left(\bar{\beta}u_1 + \tau u_3\right)\right| + |\beta| + |\tau| < \varepsilon + \|\tau\|_\infty.$$

Hence,

$$|\alpha u_2| \leqslant \left|(\bar{p}_3 - \beta u_4) - \alpha u_2\right| + \left|p_3 - \bar{\beta}u_4\right| < 2\varepsilon + \|\tau\|_\infty.$$

It follows that

$$\|u_2\|_\infty \leqslant \|\alpha^{-1}\|_\infty \varepsilon + \|\alpha^{-1}\|_\infty \|\tau\|_\infty < (6\varepsilon + m)/3m.$$

If $\varepsilon > 0$ is chosen so that $\varepsilon < m/3$, then $\|u\|_\infty < 1$. This implies that the range of $\gamma \circ u^*$ is not all of S^3. Therefore, $i(u^*) = -i(u) = [\gamma \circ u^*] = 0$, since γ is a homeomorphism. Hence if $\varepsilon < m/3$, h is an element of $\mathcal{H}_0 - \mathcal{G}_0$.

It follows from Proposition 1 that $\mathcal{H}_r - \mathcal{G}_r$ is nonempty for all r.

COROLLARY 11. *There exists a continuous extension of the analytic index on the set of Fredholm operators $\mathfrak{F}$ in $\mathfrak{B}$ to a class of operators in $\bar{\mathfrak{F}} - \mathfrak{F}$.*

PROOF. The analytic index i_{an} on $\mathfrak{F}$ is given by $\sigma \circ i$, where σ is the symbol map and i is the topological index on $\mathrm{GL}_2^{S^2 \times S^1}$. Since σ has a continuous cross section, i_{an} can be continuously extended to those operators in $\bar{\mathfrak{F}} - \mathfrak{F}$ whose symbols lie in $\mathcal{H}$.

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