Approximation by Invertibles

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Abstract. The uniform closure of the set of invertibles of certain $C^*$-algebras is characterized. It is shown that indices on $\mathcal{B}(C(A))$, where $A$ is a closed annulus in the plane and on $M_2^2\times S^1$ have continuous extensions to elements which are not invertible.

1. Introduction. It is well known that the set of invertible elements $\mathcal{G}$ in a Banach algebra is open in the algebra. Feldman and Kadison have studied the uniform closure of $\mathcal{G}$ where the algebra in question is $\mathcal{B}(\mathcal{H})$ [6]. Concerning $C^*$-algebras, in general, less is known, and we discuss this problem for certain function algebras.

A related problem is one which can be phrased very generally. Let $\mathcal{A}$ be a $C^*$-algebra with identity, and let $\mathcal{G}$ denote the set of invertibles in $\mathcal{A}$. A continuous homomorphism $\iota$ of $\mathcal{G}$ into some discrete group $\mathcal{D}$ is called an index on $\mathcal{G}$. If $r \in \mathcal{D}$, we write $\mathcal{G}_r = \iota^{-1}(r)$. For any given index we can define the set

$$\mathcal{K} = \left\{ x \in \mathcal{G} : \text{there exists an } \varepsilon > 0 \text{ and a fixed } r(x) \in \mathcal{D} : \mathcal{N}_{r(x)}(x) \cap \mathcal{G} \subset \mathcal{G}_r(x) \right\},$$

and for $r(x) \in \mathcal{D}$ we write $\mathcal{K}_r = \{ x \in \mathcal{K} : r(x) = r \}$.

We consider the problem of determining $\mathcal{K}$ for certain $C^*$-algebras whose elements are symbols of operators on a Hilbert space. Coburn and Lebow have considered the case of the usual index defined on the invertibles of the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$ where $\mathcal{H}$ is a separable Hilbert space and $\mathcal{K}$ is the set of compact operators on $\mathcal{H}$. Here $\mathcal{K} = \mathcal{G}$ [3].

The following is an immediate result of the definition.

Proposition 1. (i) If $\iota'$ is defined on $\mathcal{K}$ by $\iota'(x) = r(x)$, then $\iota' : \mathcal{K} \to \mathcal{D}$ is continuous, and $\iota'$ restricted to $\mathcal{G}$ equals $\iota$. Each $\mathcal{K}_r$ is open in $\mathcal{K}$.

(ii) $\mathcal{K}$ is not necessarily closed under products, but if $x \in \mathcal{K}_r$, $y \in \mathcal{K}_s$ and $xy \in \mathcal{K}$, then $xy \in \mathcal{K}_{r+s}$.

(iii) $\mathcal{G}_r \mathcal{G}_s \subset \mathcal{K}_{r+s}$.

(iv) $\mathcal{G}_r(\mathcal{K}_s - \mathcal{G}_s) \subset \mathcal{K}_{r+s} - \mathcal{G}_{r+s}$.

2. The algebra $\mathcal{A} = C(T)$. Let $\mathcal{A} = C(T)$ be the algebra of all complex-valued continuous functions on the unit circle $T$. A function $f \in \mathcal{A}$ is invertible if and only if $f$ does not vanish on $T$. The winding number of the
curve traced out by \( f \in \mathcal{S}(\mathbb{C}) \) with respect to origin, denoted by \( w.n.(f) \) determines an index on \( \mathcal{S} \) [2].

**Theorem 2.** Let \( i(f) = w.n.(f) \) be an index on \( \mathcal{S}(\mathbb{C}) \), and let \( \mathcal{S}_n(\mathbb{C}) = \{ f \in \mathcal{S}(\mathbb{C}) : i(f) = n \} \). Then for any \( f \in \mathcal{S}_1 \), \( f \in \bigcap_n \mathcal{S}_n(\mathbb{C}) \), so that \( \mathcal{K} = \mathcal{S} \).

**Proof.** Let \( \epsilon > 0 \) be given, and set \( S_\epsilon = \{ z \in T : |f(z)| < \epsilon \} \). Consider the function \( \phi \in C(T) \) defined as follows. For \( z \in T - S_\epsilon \), \( \phi(z) = f(z) \); for \( z = e^{i\omega} \in I_1 \), where \( I_1 = \{ e^{i\omega} : \omega_0 < \omega < \omega_1 \} \) is an arbitrary interval of \( S_\epsilon \) for which \( f(e^{i\omega_0}) = ee^{i\omega_0} \) and \( f(e^{i\theta_1}) = ee^{i\theta_1} \), let

\[
\phi(e^{i\omega}) = \epsilon \exp \left( -i \left( \frac{\omega_1 - \omega_0}{\theta_1 - \theta_0} \right) \right) \left( \omega - \omega_0 \right).
\]

The function \( \phi \in \mathcal{S}(\mathbb{C}) \), and \( \| f - \phi \| < 2\epsilon \) on \( T \). Since \( \epsilon > 0 \) is arbitrary, \( f \in \mathcal{S}(\mathbb{C}) \).

The function \( \phi \) can have arbitrary index. For if \( i(\phi) = n \) and \( k \) is any integer other than \( n \), the function \( \phi' \in \mathcal{S}(\mathbb{C}) \) defined by \( \phi'(z) = f(z) \) for \( z \in T - I_1 \), where \( I_1 = \{ e^{i\omega} : \omega_0 < \omega < \omega_1 \} \) is some fixed interval of \( S_\epsilon \), and by

\[
\phi'(z) = \begin{cases} 
\epsilon \exp \left( -i \left( \frac{\omega_1 - \omega_0}{\theta_1 - \theta_0} \right) \right) \left( \omega - \omega_0 \right), \\
\epsilon \exp i\omega_1 \exp \left( -4\pi i(k - n) \right) \left( \omega - \left( \frac{\omega_0 - \omega_1}{2} \right) \right),
\end{cases}
\]

has index \( k \), and \( \| f - \phi' \| < 2\epsilon \) on \( T \). Since \( \epsilon > 0 \) and \( k \) are arbitrary, we conclude that \( f \in \bigcap_n \mathcal{S}_n(\mathbb{C}) \).

Since every neighborhood of \( f \) contains invertibles of all indices, \( \mathcal{K} = \mathcal{S} \).

We next recall some basic notions of Fredholm theory. An operator \( A \) in \( \mathcal{B}(\mathcal{K}) \) is said to be Fredholm if the range of \( A \) is closed and \( \dim \ker A + \dim \ker A^* \) is finite. Atkinson has shown that these conditions are equivalent to \( \pi(A) \) being invertible in the algebra \( \mathcal{B}(\mathcal{K})/\mathcal{K} \) where \( \pi \) is the quotient map [1]. The analytic index \( i_\mathcal{B}(A) \) defined on the set of Fredholm operators \( \mathcal{S} \) by \( i_\mathcal{B}(A) = \dim \ker A - \dim \ker A^* \) is a homomorphism of \( \mathcal{S} \) onto the integers.

Now for \( \phi \in C(T) \), the Toeplitz operator \( T_\phi \) is defined as follows: Let \( L^2(T) \) be the space of Lebesgue square-integrable functions on \( T \) with the orthonormal basis \( \{ e^{inx}, n \in \mathbb{Z} \} \). The subspace spanned by those \( e^{inx} \) for which \( n \geq 0 \) is denoted by \( H^2(T) \). The operator \( T_\phi \) is defined on \( H^2(T) \) by \( T_\phi f = P(\phi f) \) where \( P \) is the orthogonal projection from \( L^2 \) onto \( H^2 \).

The Banach algebra \( \mathcal{B} \) generated by \( T_\phi \) has been studied by Coburn [2]. He has shown that \( \mathcal{B} = \{ T_\phi + K : \phi \in C(T), K \text{ compact} \} \) and that the \( C^* \)-algebra \( \mathcal{B}/\mathcal{K} \) is isometrically \(*\)-isomorphic to \( C(T) \). It follows that \( T_\phi \) is
a Fredholm operator if and only if $\phi$ does not vanish on $T$ and that $i_{an}(T_\phi)$ equals the negative of $\text{w.n.}(\phi)$ with respect to the origin. Lastly, let the map $\rho: \mathcal{B} \to C(T)$ be given by $\rho(T_\phi + K) = \phi$. It has been shown that the natural cross section $s$ of $\rho$ is continuous. Thus for any component $\mathcal{C}$ of $\mathcal{B}(C(T))$, $s(\mathcal{C}) = s(\mathcal{C})$ [7].

These remarks together with Theorem 2 yield the following analogue of Theorem 2 for the set of Fredholm operators $\mathcal{F}$ in $\mathcal{B}$.

**Corollary 3.** There is no possibility of continuously extending the analytic index beyond $\mathcal{F}$.

Let $\Omega$ be a compact set in the plane with interior whose boundary consists of finitely many Jordan curves. If $i$ is the index defined on $\mathcal{F}(C(\Omega))$ by $i(f) = 0$, then $\mathcal{K} = \mathcal{F}$ is always larger than $\mathcal{F}$. The problem is to determine $\mathcal{F}$. We limit ourselves to the following

**Theorem 4.** If $\Omega$ is a set of the type described, then $\mathcal{F} \subset \mathcal{K} \subset C(\Omega)$.

**Proof.** Let $a$ be an interior point of $\Omega$. The function $f(z) = z - a$ in $C(\Omega)$ has nonzero winding number along closed curves in $\Omega$ about the point $a$. Hence $f$ cannot be uniformly approximated by nonvanishing functions of $C(\Omega)$.

On the other hand, if $f \in C(\Omega)$ vanishes only on $\partial \Omega$, then $f \in \mathcal{F}$. For any component $\mathcal{C}$ of $\Omega$ with interior and any $\epsilon > 0$, let $I = \{ z \in \partial \mathcal{C} : |f(z)| < \epsilon/2 \}$. Then $I = \bigcup_i I_i$ is a countable union of disjoint smooth curves. There exist smooth arcs $\gamma_i$ in $\mathcal{C}$ having as endpoints the endpoints of $I_i$ such that on the region $R_i$ bounded by $\overline{I_i}$ and $\gamma_i$, $|f(z)| < \epsilon$. On each $R_i$ there is a continuous function $f_i$ satisfying $f_i|_{\gamma_i} = f|_{\gamma_i}$ with the range of $f_i$ on all of $R_i$ equal to the range of $f$ on $\gamma_i$. The function $f'$ defined by

$$f'(z) = \begin{cases} f(z), & z \in \mathcal{C} - \bigcup_i R_i, \\ f_i(z), & z \in R_i, \end{cases}$$

is continuous and nonvanishing on $\mathcal{C}$, and $\|f - f'\| < 2\epsilon$ on $\mathcal{C}$. Since $\mathcal{C}$ and $\epsilon > 0$ are arbitrary, $f \in \mathcal{F}$.

3. **The algebra $\mathcal{C} = C(A)$.** We now describe an algebra for which there exists a nontrivial index that can be continuously extended beyond $\mathcal{F}(\mathcal{A})$. Let $A$ be the closed annulus in the plane centered at the origin with radii $r = 1/2$ and $r = 1$. A function $f \in \mathcal{A} = C(A)$ is invertible if and only if $f$ does not vanish on $A$, and the winding number traced out by $f$ with respect to the origin along the curve $r = 1/4$ determines an index on $\mathcal{F}(\mathcal{A})$.

**Theorem 5.** For the index described, $\mathcal{F} \subset \mathcal{K} \subset \mathcal{F} \subset C(A)$.

**Proof.** If $f \in C(A)$ vanishes only on $\partial A$, then by Theorem 4, $f \in \mathcal{F}(\mathcal{A})$. The winding number of $f$ along $r = 3/4$ is some integer, say $n$. It follows from the continuity of the winding number that if $g \in \mathcal{F}(\mathcal{A})$ satisfies $\|f - g\| < \epsilon$ for $\epsilon > 0$ sufficiently small, then $\text{w.n.}(g)$ along $r = 3/4$ is also $n$. Hence, $f \in (\mathcal{K}_n - \mathcal{F}_n)$.
On the other hand, let $g \in C(A)$ be defined by $g(re^{i\theta}) = g(e^{i\theta})$, $\frac{1}{2} < r < 1$; $g(re^{i\theta'}) = 0$ for some fixed $\theta'$, and $g(re^{i\theta}) \neq 0$ for $\theta \neq \theta'$. It follows from Theorem 2 that $g \in \bigcap_n \mathcal{G}(n)$, so that $g \not\in \mathcal{H}$. Hence, $\mathcal{H} \subset \not\mathcal{G}$. Finally, it follows from Theorem 4 that if $h \in C(A)$ has an isolated zero at an interior point of $A$, then $h \not\in \not\mathcal{G}$.

The elements of $C(A)$ can be realized as symbols of operators on a Hilbert space. Because of this, we can characterize a distinguished class of non-Fredholm operators to which the analytic index can be continuously extended.

Let $\mathcal{H} = L^2(A, dA) \oplus H^2(\frac{1}{2}T, d\theta)$, where $L^2(A, dA)$ is the Hilbert space of all square-integrable functions on $A$ with respect to area measure and $H^2(\frac{1}{2}T, d\theta)$ is the Hardy space of all square-integrable functions on $\frac{1}{2}T$ with respect to normalized Haar measure. For $\phi \in C(A)$, $M_\phi$ denotes the multiplication operator on $L^2(A, dA)$ induced by $\phi$ and $T_\phi$ denotes the Toeplitz operator on $H^2(\frac{1}{2}T, d\theta)$ induced by $\phi = \phi|_{\frac{1}{2}T}$. We shall denote by $S_\phi$ the operator $M_\phi \oplus T_\phi$ on $\mathcal{H}$.

**Lemma 6.** For any $\phi, \psi \in C(A)$ the following properties hold:

(i) $S_\phi^* = S_\phi$;
(ii) $S_\lambda \phi + \mu \psi = \lambda S_\phi + \mu S_\psi$ for any complex scalars $\lambda$ and $\mu$;
(iii) $\|S_\phi\| = \|\pi(S_\phi)\| = \|\phi\|$.

**Proof.** (i) and (ii) are direct computations; (iii) is a consequence of the Douglas generalization of the Berg-Weyl theorem [5].

**Theorem 7.** The set $\mathcal{A} = \{S_\phi + K, K \in \mathcal{H}(\mathcal{H})\}$ is a Banach algebra, and the representation of elements of $\mathcal{A}$ as sums is unique.

**Proof.** It follows from the lemma that if the operators $A$ and $B$ are in $\mathcal{A}$, then so are $A + B$, $\lambda \cdot A$, and $A^*$. Since $T_\phi T_\psi - T_{\phi \psi}$ is a compact operator on $H^2(\frac{1}{2}T, d\theta)$ [2], $S_\phi S_\psi - S_{\phi \psi}$ is in $\mathcal{H}(\mathcal{H})$, and $\mathcal{A}$ is closed under multiplication. It remains to be shown that the algebra $\mathcal{A}$ is closed. By (iii) above, if a sequence $\{S_{\phi_n} + K_n\}$ is Cauchy, then so is $\{\phi_n\}$. But then $\phi_n \to \phi$ for some $\phi \in C(A)$. Again by (iii), $S_\phi \to S_\phi$ so that $K_n \to K$ for some $K \in \mathcal{H}(\mathcal{H})$, and $\mathcal{A}$ is closed. Lastly, the norm equality $\|S_\phi + K\| = \|\phi\|$ implies that the representation is unique.

**Corollary 8.** The map $S_\phi + K \leftrightarrow \phi$ is an isometric $*$-isomorphism between $\mathcal{A} / \mathcal{H}$ and $C(A)$. The operator $S_\phi$ is Fredholm in $\mathcal{A}$ if and only if $\phi$ does not vanish on $A$, and the analytic index on $S_\phi$ equals minus the winding number of $\phi$ along $r = \frac{1}{2}$ with respect to the origin.

**Proof.** The first remarks follow immediately from Proposition 6, Theorem 7, and Atkinson's characterization of Fredholm operators. Since $\text{ind } S_\phi = \text{ind } M_\phi + \text{ind } T_\phi = 0 + \text{w.n.}(\phi)$ [5], we have the index characterization of the corollary.

**Corollary 9.** The analytic index on the set of Fredholms $\not\mathcal{F} \subset \mathcal{H}$ can be continuously extended to those elements of $\mathcal{G} - \not\mathcal{G}$ whose symbols lies in $\mathcal{H}$.

**Proof.** This is an immediate consequence of Theorem 5 and the fact that the map $\phi \to S_\phi$ is continuous.
4. The algebra $\mathcal{B} = M_2^{S^2 \times S^1}$. We will consider $S^2$ as the one point compactification of the plane and $S^1$ the unit sphere $|z| = 1$. We shall study the C*-algebra $\mathcal{A} = M_2^{S^2 \times S^1}$ of $2 \times 2$ matrix-valued functions on $S^2 \times S^1$ in the matrix supremum norm. This algebra can be identified as the symbol space of the algebra $\mathcal{B}$ of $2 \times 2$ systems of singular integral operators. We briefly describe this latter algebra and refer the reader to [4] for details.

Let $\mathcal{F}f$ denote the Fourier transform of the function $f(x) \in L^2(R^2)$. The algebra $\mathcal{B}$ is generated by $2 \times 2$ systems of (a) the multiplication operators $M_\phi$, $\phi \in C(S^2)$, (b) $\mathcal{F}^{-1}M_{k(x)}\mathcal{F}$ where $k(x) \in C(S^1)$ is homogeneous of degree zero, and (c) the ideal $\mathcal{K}$ of compact operators on $L^2(R^2) \oplus L^2(R^2)$. The symbol map $\sigma$ on $\mathcal{B}$ given by

$$
\sigma(M_\phi) = \phi(x), \quad x \in S^2;
$$

$$
\sigma(\mathcal{F}^{-1}M_{k(x)}\mathcal{F}) = k(\theta), \quad \theta \in S^1;
$$

$$
\sigma(K) = 0, \quad K \text{ compact};
$$

can be extended to a *-homomorphism from all of $\mathcal{B}$ onto $M_2^{S^2 \times S^1}$ and has a continuous cross section. An operator in $\mathcal{B}$ is Fredholm if and only if its symbol has nonvanishing determinant on $S^2 \times S^1$, and two Fredholm operators are homotopic if and only if their symbols, considered as matrix functions from $S^2 \times S^1$ into $GL_2$, the general linear group of all $2 \times 2$ nonsingular matrices, are homotopic.

An index $i: \mathcal{B} = GL_2^{S^2 \times S^1} \to Z$ is determined as follows. If for $f \in \mathcal{B}$, we write $f = pu$, where $p$ is positive and

$$
\mathcal{F} \mathcal{F}^{-1} = \left( \begin{array}{cc} u_1 & u_2 \\ u_3 & u_4 \end{array} \right) \in U_2^{S^2 \times S^1},
$$

the $2 \times 2$ unitary matrices in $\mathcal{B}$, then the map $\gamma(u) = (u_1, u_2)$ determines an element of $\pi_0(S^{3\mathbb{Z} \times S^1})$. Now $\pi_0(S^{3\mathbb{Z} \times S^1}) \cong Z$, and we define $i(f) = [\gamma \circ u] \in \pi_0(S^{3\mathbb{Z} \times S^1})$, where $[\gamma \circ u]$ denotes the class of $\gamma \circ u$ in $\pi_0(S^{3\mathbb{Z} \times S^1})$ [9].

We note that it is immaterial whether the “top” or “bottom” row map is used to define the index. For

$$
\left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} \gamma & \delta \\ \alpha & \beta \end{array} \right),
$$

and $U = (0 \; 1)$ is unitary. If we write $U = e^{iH}$ for some Hermitian $H$, then $U_\epsilon = e^{i\epsilon H}$ is a homotopy between $U$ and the identity.

**Theorem 10.** For the algebra $\mathcal{B} = M_2^{S^2 \times S^1}$, where $\mathcal{B} = GL_2^{S^2 \times S^1}$ and for the index $i$ described above, $\mathcal{K} \neq \mathcal{B}$.

**Proof.** We shall describe a class of elements contained in $\mathcal{K} - \mathcal{B}$. Consider those elements in $M_2^{S^2 \times S^1}$ of the form

$$
h(x) = \left( \begin{array}{cc} \alpha(x) & \overline{\beta(x)} \\ \beta(x) & \overline{\tau(x)} \end{array} \right),
$$
where \( \alpha(x) > m > 0, \tau(x) > |\beta(x)| \) and \( m > 3\|\tau\|_\infty \). If \( \tau(x) = 0 \) on \( S^2 \times S^1 \), then \( h \not\in \mathcal{G} \). On the other hand, \( h(x) > 0 \) so that \( h \in \mathcal{G} \). Now if \( \|pu - h\| < \varepsilon \) for
\[
\begin{pmatrix} p_1 & \bar{p}_2 \\ p_2 & p_3 \end{pmatrix} \text{ positive}
\]
and
\[
\begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_4 \end{pmatrix} \text{ unitary,}
\]
then
\[
|p_3| < |p_3 - (\bar{\beta}u_1 + \tau u_3)| + |\beta| + |\tau| < \varepsilon + \|\tau\|_\infty.
\]
Hence,
\[
|au_2| < |(\bar{\beta} - \beta u_4) - au_2| + |p_3 - \bar{\beta} u_4| < 2\varepsilon + \|\tau\|_\infty.
\]
It follows that
\[
\|u_2\|_\infty \leq \|\alpha^{-1}\|_\infty \varepsilon + \|\alpha^{-1}\|_\infty \|\tau\|_\infty < (6\varepsilon + m)/3m.
\]
If \( \varepsilon > 0 \) is chosen so that \( \varepsilon < m/3 \), then \( \|u\|_\infty < 1 \). This implies that the range of \( \gamma \circ u^* \) is not all of \( S^3 \). Therefore, \( i(u^*) = -i(u) = [\gamma \circ u^*] = 0 \), since \( \gamma \) is a homeomorphism. Hence if \( \varepsilon < m/3 \), \( h \) is an element of \( \mathcal{K}_0 - \mathcal{G}_0 \).

It follows from Proposition 1 that \( \mathcal{K}_r - \mathcal{G}_r \) is nonempty for all \( r \).

**Corollary 11.** There exists a continuous extension of the analytic index on the set of Fredholm operators \( \mathcal{F} \) in \( \mathcal{G} \) to a class of operators in \( \mathcal{F} - \mathcal{G} \).

**Proof.** The analytic index \( i_{an} \) on \( \mathcal{F} \) is given by \( \sigma \circ i \), where \( \sigma \) is the symbol map and \( i \) is the topological index on \( GL_2^{S^2 \times S^1} \). Since \( \sigma \) has a continuous cross section, \( i_{an} \) can be continuously extended to those operators in \( \mathcal{F} - \mathcal{F} \) whose symbols lie in \( \mathcal{K} \).

**References**


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