WHEN MIDSETS ARE MANIFOLDS

L. D. LOVELAND

Abstract. The midset $M$ of two disjoint closed subsets $A$ and $B$ of $n$-space $E^n$ is defined as the set of all points of $E^n$ having equal distances to both $A$ and $B$. Such midsets are not always manifolds, but when either $A$ or $B$ is a convex set it follows that $M$ is homeomorphic to an open subset of an $(n - 1)$-sphere $S^{n-1}$. Furthermore, in this situation $M$ will be homeomorphic to $S^{n-1}$ if and only if the convex set $A$ is bounded and lies in the interior of the convex hull $C(B)$ of $B$. If $A$ is a singleton set and $r$ is the dimension of the smallest Euclidean flat $P$ in $E^n$ containing $A \cup B$, then $P \cap M$ is an $(r-1)$-sphere or an open $(r-1)$-cell depending upon whether or not $A$ lies in the interior (relative to $P$) of $C(B)$. In either case $M = (P \cap M) \times E^{n-r}$. A manifold lying in a midset in $E^3$ is always tamely embedded, as are $\varepsilon$-boundaries of certain special subsets of $E^n$.

1. Introduction. The midset $M(A, B)$ of two sets $A$ and $B$ in a metric space $(X, d)$ is defined to be $\{x \in X | d(x, A) = d(x, B)\}$ and is also known as the equidistant set [1], [14] or the bisector [3] of $A$ and $B$. The midset of two singleton sets in the Euclidean plane $E^2$ is the perpendicular bisector of the segment joining the two sets, and the midset of a line and a singleton set not intersecting the line is a parabola. If $A$ and $B$ are disjoint continua (nondegenerate, compact, connected sets) in $E^2$, then $M(A, B)$ is always a 1-manifold [1]; however, the analogous statement for $E^3$ is false. For a counterexample, let $A$ be the circle bounding the disk $D = \{(x, y, 0) | x^2 + y^2 < 1\}$ in $E^3$, and let $B$ be the 2-sphere $\{p \in E^3 | d(p, D) = 1\}$ where $d$ is the usual metric for $E^3$. In this example $M(A, B)$ is a "pinched" 2-sphere. We show that $M(A, B)$ is an $(n-1)$-manifold whenever $A$ and $B$ are disjoint closed subsets of $E^n$ and $A$ is a convex set. In this setting $M(A, B)$ is homeomorphic to an open subset of an $(n-1)$-sphere, and $M(A, B)$ is a topological $(n-1)$-sphere if and only if $A$ is bounded and lies in the interior of the convex hull of $B$ (see §3). A complete classification of the midsets realized when one of $A$ and $B$ is a singleton set is presented in Theorem 3.3.

One can easily prove that $M(A, B)$ is closed and that it separates $A$ from $B$ in $X$. Consequently $M(A, B)$ has dimension $n - 1$ when $X = E^n$ [12, p. 48]. Wilker [14] proved that $M(A, B)$ is connected when $A$ and $B$ are connected subsets of $E^n$. Although midsets in $E^n$ are not always manifolds, we show in
§4 that every 2-manifold lying in the midset of two disjoint closed subsets of $E^3$ must be tamely embedded in $E^3$ (Theorem 4.1).

The $\varepsilon$-boundary $\partial_\varepsilon(A)$ of a subset $A$ in $E^n$, defined as \{ $x \in E^n | d(x, A) = \varepsilon$ \} when $\varepsilon$ is a positive real number, seems somewhat related to the midset concept. Ferry [9, Theorem 3.1] proved that $\partial_\varepsilon(A)$ is a 2-manifold for almost all $\varepsilon$ if $A \subset E^3$, and he also proved a “collaring” theorem [9, Theorem 4.1].

Prior work on $\varepsilon$-boundaries was done by Brown [4], Gariepy and Pepe [11], and Weill [13]. Weill [13, p. 248] asked if a 2-sphere in $E^3$ is tamely embedded when it is realized as the $\varepsilon$-boundary of a compact set $A(\varepsilon > 0)$. In the special case where $A$ is confined to one complementary domain of $S$, Weill observed that $S$ is tame from its other complementary domain. We provide an affirmative answer to his question in the other case; that is, $S$ is tame if it is the $\varepsilon$-boundary of a set $A(\varepsilon > 0)$ that intersects both components of $E^3 - S$. In fact this result is valid in $E^n$ as long as $n \neq 4$ (Theorem 5.1).

2. Definitions and lemmas. As mentioned before, the midset of two sets $A$ and $B$ is denoted by $M(A, B)$. When no confusion results we often denote $M(A, B)$ by $M$. It is sometimes convenient to use Wilker's [14] notation \{ $A < B$ \} for the set of points in $E^n$ closer to $A$ than to $B$ and to use \{ $A \leq B$ \} to mean \{ $A < B$ \} \cup $M(A, B)$. A ray is a closed, geometric half-line in $E^n$ and is denoted sometimes by $[a, \infty)$ when $a$ is its first point. If $R$ is a ray we let $H(R)$ denote the open half-space in $E^n$ which has $R$ as its inward pointing normal and which has the first point of $R$ on its boundary. For example if $R$ is the nonnegative z-axis in $E^3$, then $H(R) = \{(x, y, z) | z > 0\}$. The boundary of $H(R)$ is an $(n - 1)$-dimensional flat. More generally a flat of dimension $r$ is a subspace of $E^n$ congruent to $E^r$.

A foot of a point $p$ on a closed set $A$ in $E^n$ is a point $\tilde{\partial}_p$ such that \(d(p, A) = d(p, \tilde{\partial}_p)\). The foot $\tilde{\partial}_p$ of a point $p$ on a convex closed set $A$ in $E^n$ is unique. A subset $A$ of $E^n$ is convex if it contains all straightline segments whose endpoints lie in $A$. The convex hull of a set $B$ in $E^n$ is the intersection of all convex sets containing $B$ and is denoted by $C(B)$.

In order to show that $M(A, B)$ is homeomorphic to an open subset of an $(n - 1)$-sphere when $A$ is convex, we use the fact that $\partial_\varepsilon(A)$ is homeomorphic to an open subset of an $(n - 1)$-sphere (Lemma 2.1), and then we identify a homeomorphism between $M(A, B)$ and an open subset of $\partial_\varepsilon(A)$ (see Theorem 3.1).

**Lemma 2.1.** Let $A$ be a closed, convex subset of $E^n$ and let $\varepsilon > 0$. Then $\partial_\varepsilon(A)$ is homeomorphic to an open subset of an $(n - 1)$-sphere. Furthermore $\partial_\varepsilon(A)$ is a topological $(n - 1)$-sphere if and only if $A$ is bounded.

**Proof.** Let $S$ be the unit $(n - 1)$-sphere centered at a point $p \in A$. For each $x \in S$, let $R_x$ be the ray $[p, \infty)$ through $x$. Such a ray intersects $\partial_\varepsilon(A)$ in at most one point as we now show. Suppose $y \in R_x \cap \partial_\varepsilon(A)$, let $\tilde{\partial}_y$ be the foot of $y$ on $A$, let $R(\tilde{\partial}_y)$, be the ray $[\tilde{\partial}_y, \infty)$ through $y$, let $H$ be the open
half-space $H(R(\partial_y))$, and let $K = E^n - H$. Since $A$ is convex, $A \cap H = \emptyset$. Thus $A \subset K$ and $y \in H$. Since $R_x$ is not parallel to the boundary flat of $H$ it is clear that $y$ is the only point of $R_x$ at a distance $\varepsilon$ from $K$. Because $A \subset K$ this means $R_x \cap \partial_x(A) = y$.

Let $D = \{x \in S | R_x \cap \partial_x(A) \neq \emptyset\}$, and define $f: D \to \partial_x(A)$ by letting $f(x)$ be the unique point of $R_x \cap \partial_x(A)$. Then $f$ is injective and the continuity of $f$ and $f^{-1}$ is easily checked.

To show $D$ is open let $x \in D$, and let $r: E^n - \{p\} \to S$ be the projection along rays emanating from $p$. The open ball $B$ of radius $\varepsilon$ and centered at $f(x)$ cannot intersect $A$, and we now show that $r(B)$, which contains $x$, lies in $D$. Let $y \in r(B)$. Since $A$ is convex and $B \cap A = \emptyset$ there must exist a subray $[y', \infty)$ of $R_y$ not intersecting $A$. From the conical structure of $r^{-1}(r(B))$ it is clear that a point $y''$ of $[y', \infty)$ exists such that the closed $\varepsilon$-ball $B'$ centered at $y''$ lies in $r^{-1}(r(B))$. Then $B' \cap A = \emptyset$ and $d(y'', A) > \varepsilon$. Thus $R_y$ intersects $\partial_x(A)$ and $y \in D$. Since $r(B) \subset D$, it follows that $D$ is open.

If $A$ is bounded, then every ray $R_x$ must intersect $\partial_x(A)$; so $f$ is surjective and $\partial_x(A)$ is homeomorphic to the sphere $S$. On the other hand, suppose $\partial_x(A)$ is an $(n-1)$-sphere. Then $A$ lies in the bounded component of $E^n - \partial_x(A)$, and $A$ is bounded.

**Lemma 2.2.** Let $A$ and $B$ be disjoint closed sets in $E^n$, and let $R$ be a ray $[a, \infty)$ with $a \in A$. If $H(R) \cap B = \emptyset$, then $R \cap M(A, B) = \emptyset$.

**Proof.** The hypothesis that $H(R) \cap B = \emptyset$ insures that for each $p \in R - \{a\}$, $d(p, B) > d(p, a) > d(p, A)$. Thus $R \subset \{A < B\}$.

**Lemma 2.3.** Let $A$ and $B$ be disjoint closed subsets of $E^n$, and let $R$ be a ray $[a, \infty)$ with $a \in A$ such that $H(R) \cap A = \emptyset$. Then $H(R) \cap B \neq \emptyset$ if and only if $R \cap M(A, B) \neq \emptyset$.

**Proof.** First we assume $R \cap M \neq \emptyset$. From the contrapositive of Lemma 2.2 it follows that $H(R) \cap B \neq \emptyset$. In the other direction it is convenient to define the continuous function $g$ taking $R$ to the real line by letting $g(t) = d(A, t) - d(t, B)$.

The hypothesis that $H(R) \cap A = \emptyset$ means $g(t) = d(a, t) - d(t, B)$, and $H(R) \cap B \neq \emptyset$ implies that, for $t$ sufficiently far from $a$, $g$ is positive. Since $g(a) < 0$, we know $g(m) = 0$ for some $m \in R$. Then $m \in R \cap M$.

**Lemma 2.4.** Let $A$ and $B$ be disjoint closed subsets of $E^n$, and let $R$ be a ray $[a, \infty)$ with $a \in A$ such that $H(R) \cap A = \emptyset$. If $R \cap M(A, B) \neq \emptyset$, then $R$ intersects $M(A, B)$ at a single point $p$. Furthermore $[a, p) \subset \{A < B\}$ and $(p, \infty) \subset \{B < A\}$.

**Proof.** Let $p$ be a point of $R \cap M$, and let $q \in R$ such that $a < q < p$. Since $d(p, a) = d(p, B)$, the ball $B(p, d(p, a))$ does not intersect $B$. However the closure of $B(q, d(q, a))$ lies in $\{a\} \cup B(p, d(p, a))$, and consequently it cannot intersect $B$. Thus $d(q, B) > d(q, a)$ and $q \in \{A < B\}$.
Similarly one can prove \( r \in \{ B < A \} \) if \( r \in R \) and \( a < p < r \).

3. Midsets are manifolds when one set is convex. In this section \( A \) and \( B \) are closed subsets of \( E^n \) and \( A \) is convex. We show that the midset \( M(A, B) \) is an \((n - 1)\)-manifold. The strategy is to construct a homeomorphism \( h \) taking \( M(A, B) \) onto an open subset of \( \partial_e(A) \) (Theorem 3.1), and then use Lemma 2.1 to conclude that \( M \) is homeomorphic to an open subset of an \((n - 1)\)-sphere. Theorem 3.2 summarizes this result and gives necessary and sufficient conditions for \( M \) to be an \((n - 1)\)-sphere.

In the special case where \( A \) is a singleton set we give an inductive classification of those manifolds which are midsets (Theorem 3.3). For example if \( A \) is a singleton and \( B \) has interior points, then \( M(A, B) \) is either an \((n - 1)\)-sphere or an open \((n - 1)\)-cell depending upon whether or not \( A \) belongs to the interior of convex hull of \( B \). A complete list of the topological types of manifolds realized as midsets when \( A \) is a singleton set and \( n = 3 \) is given in Corollary 3.2.

**Theorem 3.1.** If \( A \) and \( B \) are disjoint closed subsets of \( E^n \), \( A \) is convex, and \( \varepsilon > 0 \), then \( M(A, B) \) is homeomorphic to an open subset of \( \partial_e(A) \).

**Proof.** Let \( M = M(A, B) \), and for each point \( p \in M \) let \( \varphi_p \) be the unique foot of \( p \) on \( A \). The ray \( R(\varphi_p) \) from \( \varphi_p \) through \( p \) intersects \( \partial_e(A) \) at a unique point \( h(p) \). We now show the function \( h: M \rightarrow \partial_e(A) \) is a homeomorphism.

Suppose \( p \) and \( q \) are points of \( M \) such that \( h(p) = h(q) \). Since \( B(h(p), \varepsilon) \cap A = \emptyset \) and both \( \varphi_p \) and \( \varphi_q \) lie in the boundary of \( B(h(p), \varepsilon) \), it follows from the convexity of \( A \) that \( \varphi_p = \varphi_q \). From Lemma 2.4 we see that \( p = q \), and the injectivity of \( h \) follows. If \( \{ p_i \} \) is a sequence of points of \( M \) converging to \( p \) in \( M \), then it is clear that \( \{ \varphi_{p_i} \} \) converges to \( \varphi_p \) in \( A \). Consequently \( \{ h(p_i) \} \) converges to \( h(p) \), and \( h \) is seen to be continuous. To check the continuity of \( h^{-1} \), let \( \{ h(p_i) \} \) be a sequence converging to \( h(p) \) in \( h(M) \). Since \( d(h(p_i), \varphi_{p_i}) = \varepsilon \), for each \( i \), it is clear that \( \{ \varphi_{p_i} \} \) converges to \( \varphi_p \). Consequently every limit point of \( \{ p_i \} \) lies in the ray \( R(\varphi_p) \). But from Lemma 2.4, \( R(\varphi_p) \cap M = \{ p \} \). Thus \( \{ p_i \} \) converges to \( p \), as desired.

Suppose \( h(M) \) is not open in \( \partial_e(A) \). Then there is a sequence \( \{ x_i \} \) of points of \( \partial_e(A) - h(M) \) converging to a point \( h(p) \) of \( h(M) \). For each \( i \) let \( \varphi_{x_i} \) be the unique foot of \( x_i \) on \( A \), and let \( R_i \) be the ray \([\varphi_{x_i}, \infty) \) through \( x_i \). Since \( R_i \cap M = \emptyset \), we see that \( R_i \subset \{ A < B \} \) for each \( i \). But \( \{ R_i \} \) converges to \( R(\varphi_p) \), so \( R(\varphi_p) \subset \{ A < B \} \), contradicting Lemma 2.4. Thus \( h(M) \) is an open subset of \( \partial_e(A) \).

**Theorem 3.2.** Let \( A \) and \( B \) be disjoint closed subsets of \( E^n \) with \( A \) convex. Then \( M(A, B) \) is homeomorphic to an open subset of an \((n - 1)\)-sphere, and \( M(A, B) \) is a topological \((n - 1)\)-sphere if and only if \( A \) is bounded and lies in \( \text{Int} \ C(B) \).
Proof. Let \( h : M(A, B) \rightarrow \partial_x(A) \) be the homeomorphism defined in the proof of Theorem 3.1. From Lemma 2.1 it follows that \( h \) takes \( M \) onto an open subset of the \((n - 1)\)-sphere \( \partial_x(A) \). To establish the necessary and sufficient condition for \( M \) to be an \((n - 1)\)-sphere, let us first assume \( M \) is such a sphere. Of course it follows that \( \partial_x(A) \) is an \((n - 1)\)-sphere, and from Lemma 2.1, \( A \) must be bounded. Suppose \( A \notin \text{Int} \ C(B) \). Then a ray \( R \) emanating from a point \( a' \in A \) exists such that \( H(R) \cap B = \emptyset \). Since \( A \) is compact, \( R \) may be chosen so that \( H(R) \cap A = \emptyset \). Then it follows from Lemma 2.3 that \( R \cap M = \emptyset \). But this is a contradiction since \( R \) must intersect \( \partial_x(A) \) at some point \( h(p) \) and the definition of \( h \) requires that \( R \) intersect \( M \) at some point \( h(p) \). Thus \( A \subset \text{Int} \ C(B) \).

Now we assume \( A \) is a bounded subset of \( \text{Int} \ C(B) \). From Lemma 2.1 \( \partial_x(A) \) is an \((n - 1)\)-sphere, so it suffices to prove \( h \) surjective. Let \( x \in \partial_x(A) \), let \( \partial_x \) be the unique foot of \( x \) on \( A \), and let \( R \) be the ray \([a_x, \infty) \) through \( x \). Then \( H(R) \cap B \neq \emptyset \) since \( A \subset \text{Int} \ C(B) \). Since \( H(R) \cap A = \emptyset \) it follows from Lemma 2.4 that \( R \cap A \) contains a point \( \beta \). Then, by the definition of \( h \), \( h(\beta) = x \). Since \( h \) is surjective, \( M \) is an \((n - 1)\)-sphere.

Corollary 3.1. Let \( A \) and \( B \) be disjoint subsets of \( E^n \) with \( A \) convex, and let \( P \) be the flat of least dimension containing \( A \cup B \). If \( P \) has dimension \( r \), then \( P \cap M(A, B) \) is an \((r - 1)\)-manifold \( K \) homeomorphic to an open subset of an \((r - 1)\)-sphere and \( M(A, B) \) is homeomorphic to \( K \times E^{n-r} \).

Theorem 3.3. Let \( A \) and \( B \) be disjoint closed subsets of \( E^n \) where \( A \) is the singleton set \( \{a\} \), let \( P \) be the flat of least dimension containing \( \{a\} \cup B \), and let \( \text{Int}_p C(B) \) denote the interior of the convex hull of \( B \) relative to the space \( P \). If \( P \) has dimension \( r \), then

(i) \( P \cap M(A, B) \) is an \((r - 1)\)-sphere if and only if \( a \in \text{Int}_p C(B) \),
(ii) \( P \cap M(A, B) \) is an open \((r - 1)\)-cell if and only if \( a \notin \text{Int}_p C(B) \), and
(iii) in either case, \( M(A, B) = (P \cap M(A, B)) \times E^{n-r} \).

Proof. For convenience in the notation we let \( M \) denote \( P \cap M(A, B) \). Since \( A \) is convex, conclusion (i) and one direction of conclusion (ii) follow from Theorem 3.2. Since conclusion (iii) is clear, it remains to show that \( M \) is an open \((r - 1)\)-cell if \( a \notin \text{Int}_p C(B) \).

Let \( S \) be an \((r - 1)\)-sphere in \( P \) centered at \( a \), and let \( \pi : M \rightarrow S \) be the projection along rays emanating from \( a \). Since \( S \) is an \( \varepsilon \)-boundary of \( A \), previously given proofs show that \( \pi \) is a homeomorphism of \( M \) onto an open \((r - 1)\)-manifold subset of \( S \). The complement \( X \) of \( \pi(M) \) in \( S \) is not empty since \( a \notin \text{Int}_p C(B) \). From Lemma 2.3 and the fact that \( P \) is the minimal flat containing \( \{a\} \cup b \), we see that the closed set \( X \) contains no antipodal points of \( S \). From this fact and Lemma 2.3 it follows that \( X \) is 'convex' in the sense that it contains, with each pair of its points, the smaller arc of the great circle determined by them. These two facts imply that \( X \) is cellular (using the technique in the proof of Lemma 2.1, one can construct a sequence \( \{C_i\} \) of
(r - 1)-cells in S such that \(X = \cap_{i=1}^{n} C_i\) and, for each \(i\), \(X \subset \text{Int} C_i \subset C_i \subset \text{Int} C_{i+1}\); consequently \(S - X\) is homeomorphic to an open \((r - 1)\)-cell.

**Corollary 3.2.** If A and B are disjoint closed subsets of \(E^3\) and A is a singleton set, then \(M(A, B)\) is homeomorphic to either a 2-sphere, an open 2-cell, an open annulus, or to the union of two disjoint open 2-cells.

**Corollary 3.3.** If A and B are disjoint closed subsets of \(E^n\) and A is convex, then the following statements are equivalent:

1. \(M(A, B)\) is an \((n - 1)\)-sphere.
2. A is bounded and lies in \(\text{Int} C(B)\).
3. \(M(A, B)\) is bounded.

**Proof.** Statements (1) and (2) are equivalent by Theorem 3.2, and (1) implies (3). Suppose \(M\) is bounded. Since \(M\) separates A from B and is homeomorphic to an open subset of an \((n - 1)\)-sphere (Theorem 3.2), \(M\) must be an \((n - 1)\)-sphere. Thus (3) implies (1).

**Remarks and Examples.** When dealing with the midset \(M\) of two connected subsets of \(E^n\) it is important to realize that \(M\) is connected \([14]\). Thus when B is connected in Corollary 3.2, there are only three possibilities for the midset.

An example was given in §1 of a nonmanifold midset \(M(A, B)\) in \(E^3\) where A is a circle and B is a 2-sphere. Of course A is not convex nor is it simply connected. However B is simply connected, so this condition is not sufficient to insure a manifold midset. We leave to the interested reader the description of arcs A and B in \(E^3\) whose midset is not a manifold.

A classification of all midset manifolds resulting when A is convex would be interesting. Theorem 3.3 gives the situation when A is a singleton set. A natural attempt at such a classification is to prove that a point a can always be chosen in A such that \(M(A, B)\) is homeomorphic to \(M(\{a\}, B)\), but this is not generally true. In \(E^2\), let \(A = \{(x, y) \mid y < 2\text{ and } x = 0\}\) and let \(B = \{(x, y) \mid x = 1\text{ and } |y| < 1\}\). Then no such point in A exists. Nevertheless, the manifolds listed in Corollary 3.2 seem to be the only ones that are realized in \(E^3\) even when the convex set A is not a singleton set.

**4. Tame midset manifolds.** In the situation where \(A = \{a\}\) and \(a \notin B\), it is easy to prove the resulting \((n - 1)\)-manifold \(M(A, B)\) tame in \(E^3\) because the radial map \(h\) defined in the proof of Theorem 3.1 can be extended to a homeomorphism from a bicollar on \(M(A, B)\) to one on the \((n - 1)\)-sphere \(\partial_r(A)\). The general situation where neither set is degenerate seems more difficult; nevertheless when midsets are manifolds in \(E^3\) they must be tame.

**Theorem 4.1.** If a 2-manifold \(K\) is a subset of the midset \(M\) of two disjoint closed subsets A and B of \(E^3\), then \(K\) is tame in \(E^3\).

**Proof.** From the hypothesis, \(M\) is the common boundary of the disjoint open sets \(\{A < B\}\) and \(\{B < A\}\) (see [14, Theorem 2]). Let \(m \in K\) and let \(a\)
be a foot of \( m \) on \( A \). The set \( \{a \leq B\} \) is convex because it is the intersection of the collection \( \{\{a \leq \{b\}\}\mid b \in B\} \) of closed, convex, half-spaces. Since \( \{a \leq B\} \) contains \( m \), is convex, and has \( a \) in its interior, it contains a solid right circular cone \( C_m \) with vertex \( m \). The cone \( C_m \) lies in \( \{A \leq B\} \) because \( \{a \leq B\} \) does. Hence \( K \) can be "touched by a pencil" at each \( m \in K \) from \( \{A \leq B\} \). Similarly such a cone exists in \( \{B \leq A\} \).

It follows directly from Corollary 4.6 of [5] that \( K \) is tame if \( \Gamma \) is a 2-sphere. Otherwise, for each \( m \in K \), there exist a 2-sphere \( S \) and a disk \( D \) in \( K \cap S \) with \( m \in \text{Int} \ D \) (see [2, Theorem 5.4.1, p. 296]). For convenience we assume \( D \) lies in the closure of \( (\text{Int} S) \cap \{A \leq B\} \). Let \( E \) be a disk such that \( M \in \text{Int} E \subset E \subset \text{Int} D \), and for each \( p \in E \) choose a subcone \( K_p \) of \( C_p \) small enough to lie in \( S \cup \text{Int} S \). Let \( X \) be the closure of \( \bigcup_{p \in E} K_p \), and notice that \( X \) is the union of convex 3-cells. Adjust \( S \) to obtain a 2-sphere \( S' \) such that \( X \subset S' \cup \text{Int} S' \) and \( S' \) is locally tame modulo \( X \) (see [2, Theorem 4.6.1, p. 282]). By Theorem 4.5 of [5], \( S' \) is tame from \( \text{Ext} S' \), which implies that \( K \) is locally tame at \( m \) from \( \{B \leq A\} \). A similar argument, where the cones \( K_p \) are selected in \( S \cup \text{Ext} S \), shows \( K \) locally tame at \( m \) from \( \{A \leq B\} \); hence \( K \) is tame.

Remarks. The above proof generalizes to \( E^n(n > 3) \) to show that an \((n - 1)-\text{manifold subset of the midset of disjoint closed sets } A \text{ and } B \) can be "touched by convex n-cells" from both \( \{A \leq B\} \) and \( \{A \geq B\} \); however, this does not imply tameness for an \((n - 1)-\text{manifold in } E^n \) if \( n > 4 \). An example can be obtained using Daverman's "inflation" technique [8, §11].

5. When an \( \epsilon \)-boundary is tame. Weill [13, p. 248] asked if a 2-sphere \( S \) in \( E^3 \) is tame when it is, for some \( \epsilon > 0 \), the \( \epsilon \)-boundary of a compact set \( X \). As he pointed out, the special case where \( X \) lies in one component \( U \) of \( E^3 - S \) can be partially resolved using Theorem 4.5 of [5]; in this case \( S \) is tame from \( E^3 - (S \cup U) \). However \( S \) must be tame from both sides in the case where \( X \) intersects both \( \text{Int} S \) and \( \text{Ext} S \); in fact, in this case the result carries over to \( E^n(n \neq 4) \).

Theorem 5.1. If a compact connected \((n - 1)-\text{manifold } M \) in \( E^n \) is the \( \epsilon \)-boundary of a set \( A \) that intersects both components of \( E^n - M \), then \( M \) is tame in \( E^n(n \neq 4) \).

Proof. We may assume \( A \) closed since \( \partial_{\epsilon}(A) = \partial_{\epsilon}(\overline{A}) \). Define \( E = A \cap \text{Ext} M \) and \( I = A \cap \text{Int} M \). To show that \( \partial_{\epsilon}(E) = M \) suppose \( p \in \partial_{\epsilon}(E) - M \). Since \( p \not\in \partial_{\epsilon}(A) \), we have \( d(p, I) < \epsilon \). But then \( M \) fails to separate \( E \) from \( I \) because there exist segments \([a, p]\) and \([p, e]\) whose union is a path from \( a \in I \) to \( e \in E \) which fails to intersect \( M \). Thus \( \partial_{\epsilon}(E) \subset M \), and it follows that \( \partial_{\epsilon}(E) = M \) since \( \partial_{\epsilon}(E) \) must separate \( E^3 \). Similarly \( \partial_{\epsilon}(I) = M \).

In case \( n = 3 \), Theorem 5.1 can be reduced to Theorem 4.1 by showing that \( M \) is the midset of \( I \) and \( E \); however, we do not make this restriction on \( n \). The strategy is to show that \( E^n - M \) is \( 1-\text{ULC} \), for then the tameness of \( M \) follows from [6] or [7] if \( n > 5 \) and from Bing's result [2, Theorem 6.4.1] if
$n = 3$. To show $E^n - M$ is $1 - \text{ULC}$ we first note that for each point $p \in M$ there are points $a$ and $b$ of $I$ and $E$, respectively, such that the closed balls $B(a, \varepsilon)$ and $B(b, \varepsilon)$ are tangent at $p$ and lie in the closures of $\text{Int} M$ and $\text{Ext} M$, respectively. Griffith [10] proved that such a manifold must be locally spanned from both complementary domains, and Burgess [2, Theorem 6.5.2, p. 305] showed that locally spanned manifolds have $1 - \text{ULC}$ complementary domains. Although the Burgess and Griffith results just cited were stated for $E^3$ the proofs work as well in higher dimensions.

**Remarks.** Roughly speaking, Theorem 5.1 says that a manifold $M$ is tame when it is the $\varepsilon$-boundary of a set $A$ which has two pieces separated by a gap of width $2\varepsilon$. One can extract a seemingly more general theorem to the effect that a manifold $\varepsilon$-boundary $\partial_\varepsilon(A)$ is tame provided it is the mindset of $A$ and $\partial_{2\varepsilon}(A)$. However this situation does not seem to arise except in obvious cases (for example when $A$ is convex) in which the tameness of $\partial_\varepsilon(A)$ is clear. In this connection it is interesting to let $A$ be the union of unit intervals $[0, 0)$, $(1,0)]$ and $[0, (0, 1)]$ in $E^2$. There is no positive number $\varepsilon$ such that $\partial_\varepsilon(A) = M(A, \partial_{2\varepsilon}(A))$, yet $A$ is a rather simple set.

**References**

1. H. Bell, *Some topological extensions of plane geometry* (manuscript).