PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 2, December 1976

WHEN MIDSETS ARE MANIFOLDS

L. D. LOVELAND

ABSTRACT. The midset M of two disjoint closed subsets A and B of n-space E^n is defined as the set of all points of E^n having equal distances to both A and B. Such midsets are not always manifolds, but when either A or B is a convex set it follows that M is homeomorphic to an open subset of an $(n-1)$-sphere S^{n-1}. Furthermore, in this situation M will be homeomorphic to S^{n-1} if and only if the convex set A is bounded and lies in the interior of the convex hull $C(B)$ of B. If A is a singleton set and r is the dimension of the smallest Euclidean flat P in E^n containing $A \cup B$, then $P \cap M$ is an $(r-1)$-sphere or an open $(r-1)$-cell depending upon whether or not A lies in the interior (relative to P) of $C(B)$. In either case $M = (P \cap M) \times E^{n-r}$. A manifold lying in a midset in E^3 is always tamely embedded, as are ε-boundaries of certain special subsets of E^n.

1. **Introduction.** The *midset* $M(A, B)$ of two sets A and B in a metric space (X, d) is defined to be $\{x \in X | d(x, A) = d(x, B)\}$ and is also known as the *equidistant set* [1], [14] or the *bisector* [3] of A and B. The midset of two singleton sets in the Euclidean plane E^2 is the perpendicular bisector of the segment joining the two sets, and the midset of a line and a singleton set not intersecting the line is a parabola. If A and B are disjoint continua (nondegenerate, compact, connected sets) in E^2, then $M(A, B)$ is always a 1-manifold [1]; however, the analogous statement for E^3 is false. For a counterexample, let A be the circle bounding the disk $D = \{(x, y, 0) | x^2 + y^2 \leqslant 1\}$ in E^3, and let B be the 2-sphere $\{p \in E^3 | d(p,D) = 1\}$ where d is the usual metric for E^3. In this example $M(A, B)$ is a "pinched" 2-sphere. We show that $M(A, B)$ is an $(n-1)$-manifold whenever A and B are disjoint closed subsets of E^n and A is a convex set. In this setting $M(A, B)$ is homeomorphic to an open subset of an $(n-1)$-sphere, and $M(A, B)$ is a topological $(n-1)$-sphere if and only if A is bounded and lies in the interior of the convex hull of B (see §3). A complete classification of the midsets realized when one of A and B is a singleton set is presented in Theorem 3.3.

One can easily prove that $M(A, B)$ is closed and that it separates A from B in X. Consequently $M(A, B)$ has dimension $n-1$ when $X = E^n$ [12, p. 48]. Wilker [14] proved that $M(A, B)$ is connected when A and B are connected subsets of E^n. Although midsets in E^n are not always manifolds, we show in

Received by the editors January 5, 1976 and, in revised form, April 19, 1976.

AMS (*MOS*) *subject classifications* (1970). Primary 50B99, 54F99; Secondary 52A20, 54E35, 57A10, 57A15.

Key words and phrases. Bisector, ε-boundary, equidistant set, Euclidean n-space, manifold midset, midsets, tame manifolds, tame surfaces.

§4 that every 2-manifold lying in the midset of two disjoint closed subsets of E^3 must be tamely embedded in E^3 (Theorem 4.1).

The *ε-boundary* $\partial_\varepsilon(A)$ of a subset A in E^n, defined as $\{x \in E^n | d(x, A) = \varepsilon\}$ when ε is a positive real number, seems somewhat related to the midset concept. Ferry [9, Theorem 3.1] proved that $\partial_\varepsilon(A)$ is a 2-manifold for almost all ε if $A \subset E^3$, and he also proved a "collaring" theorem [9, Theorem 4.1]. Prior work on ε-boundaries was done by Brown [4], Gariepy and Pepe [11], and Weill [13]. Weill [13, p. 248] asked if a 2-sphere in E^3 is tamely embedded when it is realized as the ε-boundary of a compact set $A(\varepsilon > 0)$. In the special case where A is confined to one complementary domain of S, Weill observed that S is tame from its other complementary domain. We provide an affirmative answer to his question in the other case; that is, S is tame if it is the ε-boundary of a set $A(\varepsilon > 0)$ that intersects both components of $E^3 - S$. In fact this result is valid in E^n as long as $n \neq 4$ (Theorem 5.1).

2. **Definitions and lemmas.** As mentioned before, the midset of two sets A and B is denoted by $M(A, B)$. When no confusion results we often denote $M(A, B)$ by M. It is sometimes convenient to use Wilker's [14] notation $\{A < B\}$ for the set of points in E^n closer to A than to B and to use $\{A \leqslant B\}$ to mean $\{A < B\} \cup M(A, B)$. A *ray* is a closed, geometric half-line in E^n and is denoted sometimes by $[a, \infty)$ when a is its first point. If R is a ray we let $H(R)$ denote the open half-space in E^n which has R as its inward pointing normal and which has the first point of R on its boundary. For example if R is the nonnegative z-axis in E^3, then $H(R) = \{(x, y, z) | z > 0\}$. The boundary of $H(R)$ is an $(n - 1)$-dimensional flat. More generally a *flat* of dimension r is a subspace of E^n congruent to E^r.

A *foot* of a point p on a closed set A in E^n is a point $\mathfrak{F}_p$ such that $d(p, A) = d(p, \mathfrak{F}_p)$. The foot $\mathfrak{F}_p$ of a point p on a convex closed set A in E^n is unique. A subset A of E^n is *convex* if it contains all straightline segments whose endpoints lie in A. The *convex hull* of a set B in E^n is the intersection of all convex sets containing B and is denoted by $C(B)$.

In order to show that $M(A, B)$ is homeomorphic to an open subset of an $(n - 1)$-sphere when A is convex, we use the fact that $\partial_\varepsilon(A)$ is homeomorphic to an open subset of an $(n - 1)$-sphere (Lemma 2.1), and then we identify a homeomorphism between $M(A, B)$ and an open subset of $\partial_\varepsilon(A)$ (see Theorem 3.1).

LEMMA 2.1. *Let A be a closed, convex subset of E^n and let $\varepsilon > 0$. Then $\partial_\varepsilon(A)$ is homeomorphic to an open subset of an $(n - 1)$-sphere. Furthermore $\partial_\varepsilon(A)$ is a topological $(n - 1)$-sphere if and only if A is bounded.*

PROOF. Let S be the unit $(n - 1)$-sphere centered at a point $p \in A$. For each $x \in S$, let R_x be the ray $[p, \infty)$ through x. Such a ray intersects $\partial_\varepsilon(A)$ in at most one point as we now show. Suppose $y \in R_x \cap \partial_\varepsilon(A)$, let $\mathfrak{F}_y$ be the foot of y on A, let $R(\mathfrak{F}_y)$, be the ray $[\mathfrak{F}_y, \infty)$ through y, let H be the open half-space $H(R(\mathcal{C}_y))$, and let $K = E^n - H$. Since A is convex, $A \cap H = \varnothing$. Thus $A \subset K$ and $y \in H$. Since R_x is not parallel to the boundary flat of H it is clear that y is the only point of R_x at a distance ε from K. Because $A \subset K$ this means $R_x \cap \partial_\varepsilon(A) = y$.

Let $D = \{x \in S | R_x \cap \partial_\varepsilon(A) \neq \varnothing\}$, and define f: $D \to \partial_\varepsilon(A)$ by letting $f(x)$ be the unique point of $R_x \cap \partial_\varepsilon(A)$. Then f is injective and the continuity of f and f^{-1} is easily checked.

To show D is open let $x \in D$, and let r: $E^n - \{p\} \to S$ be the projection along rays emanating from p. The open ball B of radius ε and centered at $f(x)$ cannot intersect A, and we now show that $r(B)$, which contains x, lies in D. Let $y \in r(B)$. Since A is convex and $B \cap A = \varnothing$ there must exist a subray $[y', \infty)$ of R_y not intersecting A. From the conical structure of $r^{-1}(r(B))$ it is clear that a point y'' of $[y', \infty)$ exists such that the closed ε-ball B' centered at y'' lies in $r^{-1}(r(B))$. Then $B' \cap A = \varnothing$ and $d(y'', A) > \varepsilon$. Thus R_y intersects $\partial_\varepsilon(A)$ and $y \in D$. Since $r(B) \subset D$, it follows that D is open.

If A is bounded, then every ray R_x must intersect $\partial_\varepsilon(A)$; so f is surjective and $\partial_\varepsilon(A)$ is homeomorphic to the sphere S. On the other hand, suppose $\partial_\varepsilon(A)$ is an $(n-1)$-sphere. Then A lies in the bounded component of $E^n - \partial_\varepsilon(A)$, and A is bounded.

LEMMA 2.2. *Let A and B be disjoint closed sets in E^n, and let R be a ray $[a, \infty)$ with $a \in A$. If $H(R) \cap B = \varnothing$, then $R \cap M(A, B) = \varnothing$.*

PROOF. The hypothesis that $H(R) \cap B = \varnothing$ insures that for each $p \in R - \{a\}$, $d(p, B) > d(p, a) \geq d(p, A)$. Thus $R \subset \{A < B\}$.

LEMMA 2.3. *Let A and B be disjoint closed subsets of E^n, and let R be a ray $[a, \infty)$ with $a \in A$ such that $H(R) \cap A = \varnothing$. Then $H(R) \cap B \neq \varnothing$ if and only if $R \cap M(A, B) \neq \varnothing$.*

PROOF. First we assume $R \cap M \neq \varnothing$. From the contrapositive of Lemma 2.2 it follows that $H(R) \cap B \neq \varnothing$. In the other direction it is convenient to define the continuous function g taking R to the real line by letting $g(t) = d(A, t) - d(t, B)$.The hypothesis that $H(R) \cap A = \varnothing$ means $g(t) = d(a, t) - d(t, B)$, and $H(R) \cap B \neq \varnothing$ implies that, for t sufficiently far from a, g is positive. Since $g(a) < 0$, we know $g(m) = 0$ for some $m \in R$. Then $m \in R \cap M$.

LEMMA 2.4. *Let A and B be disjoint closed subsets of E^n, and let R be a ray $[a, \infty)$ with $a \in A$ such that $H(R) \cap A = \varnothing$. If $R \cap M(A, B) \neq \varnothing$, then R intersects $M(A, B)$ at a single point p. Furthermore $[a, p) \subset \{A < B\}$ and $(p, \infty) \subset \{B < A\}$.*

PROOF. Let p be a point of $R \cap M$, and let $q \in R$ such that $a < q < p$. Since $d(p, a) = d(p, B)$, the ball $B(p, d(p, a))$ does not intersect B. However the closure of $B(q, d(q, a))$ lies in $\{a\} \cup B(p, d(p, a))$, and consequently it cannot intersect B. Thus $d(q, B) > d(q, a)$ and $q \in \{A < B\}$.

Similarly one can prove $r \in \{B < A\}$ if $r \in R$ and $a < p < r$.

3. **Midsets are manifolds when one set is convex.** In this section A and B are closed subsets of E^n and A is convex. We show that the midset $M(A, B)$ is an $(n-1)$-manifold. The strategy is to construct a homeomorphism h taking $M(A, B)$ onto an open subset of $\partial_\varepsilon(A)$ (Theorem 3.1), and then use Lemma 2.1 to conclude that M is homeomorphic to an open subset of an $(n-1)$-sphere. Theorem 3.2 summarizes this result and gives necessary and sufficient conditions for M to be an $(n-1)$-sphere.

In the special case where A is a singleton set we give an inductive classification of those manifolds which are midsets (Theorem 3.3). For example if A is a singleton and B has interior points, then $M(A, B)$ is either an $(n-1)$-sphere or an open $(n-1)$-cell depending upon whether or not A belongs to the interior of convex hull of B. A complete list of the topological types of manifolds realized as midsets when A is a singleton set and $n = 3$ is given in Corollary 3.2.

THEOREM 3.1. *If A and B are disjoint closed subsets of E^n, A is convex, and $\varepsilon > 0$, then $M(A, B)$ is homeomorphic to an open subset of $\partial_\varepsilon(A)$.*

PROOF. Let $M = M(A, B)$, and for each point $p \in M$ let $\mathfrak{a}_p$ be the unique foot of p on A. The ray $R(\mathfrak{a}_p)$ from $\mathfrak{a}_p$ through p intersects $\partial_\varepsilon(A)$ at a unique point $h(p)$. We now show the function $h: M \to \partial_\varepsilon(A)$ is a homeomorphism.

Suppose p and q are points of M such that $h(p) = h(q)$. Since $B(h(p), \varepsilon) \cap A = \varnothing$ and both $\mathfrak{a}_p$ and $\mathfrak{a}_q$ lie in the boundary of $B(h(p), \varepsilon)$, it follows from the convexity of A that $\mathfrak{a}_p = \mathfrak{a}_q$. From Lemma 2.4 we see that $p = q$, and the injectivity of h follows. If $\{p_i\}$ is a seqquence of points of M converging to p in M, then it is clear that $\{\mathfrak{a}_{p_i}\}$ converges to $\mathfrak{a}_p$ in A. Consequently $\{h(p_i)\}$ converges to $h(p)$, and h is seen to be continuous. To check the continuity of h^{-1}, let $\{h(p_i)\}$ be a sequence converging to $h(p)$ in $h(M)$. Since $d(h(p_i), \mathfrak{a}_{p_i}) = \varepsilon$, for each i, it is clear that $\{\mathfrak{a}_{p_i}\}$ converges to $\mathfrak{a}_p$. Consequently every limit point of $\{p_i | i = 1\ 2, 3, \ldots\}$ lies in the ray $R(\mathfrak{a}_p)$. But from Lemma 2.4, $R(\mathfrak{a}_p) \cap M = \{p\}$. Thus $\{p_i\}$ converges to p, as desired.

Suppose $h(M)$ is not open in $\partial_\varepsilon(A)$. Then there is a sequence $\{x_i\}$ of points of $\partial_\varepsilon(A) - h(M)$ converging to a point $h(p)$ of $h(M)$. For each i let $\mathfrak{a}_i$ be the unique foot of x_i on A, and let R_i be the ray $[\mathfrak{a}_i, \infty)$ through x_i. Since $R_i \cap M = \varnothing$, we see that $R_i \subset \{A < B\}$ for each i. But $\{R_i\}$ converges to $R(\mathfrak{a}_p)$, so $R(\mathfrak{a}_p) \subset \{A \leqslant B\}$, contradicting Lemma 2.4. Thus $h(M)$ is an open subset of $\partial_\varepsilon(A)$.

THEOREM 3.2. *Let A and B be disjoint closed subsets of E^n with A convex. Then $M(A, B)$ is homeomorphic to an open subset of an $(n-1)$-sphere, and $M(A, B)$ is a topological $(n-1)$-sphere if and only if A is bounded and lies in* Int $C(B)$.

PROOF. Let h: $M(A, B) \to \partial_\varepsilon(A)$ be the homeomorphism defined in the proof of Theorem 3.1. From Lemma 2.1 it follows that h takes M onto an open subset of the $(n-1)$-sphere $\partial_\varepsilon(A)$. To establish the necessary and sufficient condition for M to be an $(n-1)$-sphere, let us first assume M is such a sphere. Of course it follows that $\partial_\varepsilon(A)$ is an $(n-1)$-sphere, and from Lemma 2.1, A must be bounded. Suppose $A \not\subset \text{Int } C(B)$. Then a ray R emanating from a point $a' \in A$ exists such that $H(R) \cap B = \varnothing$. Since A is compact, R may be chosen so that $H(R) \cap A = \varnothing$. Then it follows from Lemma 2.3 that $R \cap M = \varnothing$. But this is a contradiction since R must intersect $\partial_\varepsilon(A)$ at some point $h(p)$ and the definition of h requires that R intersect M at p. Thus $A \subset \text{Int } C(B)$.

Now we assume A is a bounded subset of Int $C(B)$. From Lemma 2.1 $\partial_\varepsilon(A)$ is an $(n-1)$-sphere, so it suffices to prove h surjective. Let $x \in \partial_\varepsilon(A)$, let $\mathcal{Q}_x$ be the unique foot of x on A,and let R be the ray $[a_x, \infty)$ through x. Then $H(R) \cap B \neq \varnothing$ since $A \subset \text{Int } C(B)$. Since $H(R) \cap A = \varnothing$ it follows from Lemma 2.4 that $R \cap M$ contains a point p. Then, by the definition of h, $h(p) = x$. Since h is surjective, M is an $(n-1)$-sphere.

COROLLARY 3.1. *Let A and B be disjoint subsets of E^n with A convex, and let P be the flat of least dimension containing $A \cup B$. If P has dimension r, then $P \cap M(A, B)$ is an $(r-1)$-manifold K homeomorphic to an open subset of an $(r-1)$-sphere and $M(A, B)$ is homeomorphic to $K \times E^{n-r}$.*

THEOREM 3.3. *Let A and B be disjoint closed subsets of E^n where A is the singleton set $\{a\}$, let P be the flat of least dimension containing $\{a\} \cup B$, and let $\text{Int}_P\, C(B)$ denote the interior of the convex hull of B relative to the space P. If P has dimension r, then*

(i) *$P \cap M(A, B)$ is an $(r-1)$-sphere if and only if $a \in \text{Int}_P C(B)$,*
(ii) *$P \cap M(A, B)$ is an open $(r-1)$-cell if and only if $a \notin \text{Int}_P C(B)$, and*
(iii) *in either case, $M(A, B) = (P \cap M(A, B)) \times E^{n-r}$.*

PROOF. For convenience in the notation we let M denote $P \cap M(A, B)$. Since A is convex, conclusion (i) and one direction of conclusion (ii) follow from Theorem 3.2. Since conclusion (iii) is clear, it remains to show that M is an open $(r-1)$-cell if $a \notin \text{Int}_P C(B)$.

Let S be an $(r-1)$-sphere in P centered at a, and let π: $M \to S$ be the projection along rays emanating from a. Since S is an ε-boundary of A, previously given proofs show that π is a homeomorphism of M onto an open $(r-1)$-manifold subset of S. The complement X of $\pi(M)$ in S is not empty since $a \notin \text{Int}_P C(B)$. From Lemma 2.3 and the fact that P is the minimal flat containing $\{a\} \cup b$, we see that the closed set X contains no antipodal points of S. From this fact and Lemma 2.3 it follows that X is 'convex' in the sense that it contains, with each pair of its points, the smaller arc of the great circle determined by them. These two facts imply that X is cellular (using the technique in the proof of Lemma 2.1, one can construct a sequence $\{C_i\}$ of

$(r-1)$-cells in S such that $X = \bigcap_{i=1}^{\infty} C_i$ and, for each i, $X \subset \text{Int } C_i \subset C_i \subset \text{Int } C_{i+1}$); consequently $S - X$ is homeomorphic to an open $(r-1)$-cell.

COROLLARY 3.2. *If A and B are disjoint closed subsets of E^3 and A is a singleton set, then $M(A, B)$ is homeomorphic to either a 2-sphere, an open 2-cell, an open annulus, or to the union of two disjoint open 2-cells.*

COROLLARY 3.3. *If A and B are disjoint closed subsets of E^n and A is convex, then the following statements are equivalent*:

(1) *$M(A, B)$ is an $(n-1)$-sphere.*

(2) *A is bounded and lies in* Int $C(B)$.

(3) *$M(A, B)$ is bounded.*

PROOF. Statements (1) and (2) are equivalent by Theorem 3.2, and (1) implies (3). Suppose M is bounded. Since M separates A from B and is homeomorphic to an open subset of an $(n-1)$-sphere (Theorem 3.2), M must be an $(n-1)$-sphere. Thus (3) implies (1).

REMARKS AND EXAMPLES. When dealing with the midset M of two connected subsets of E^n it is important to realize that M is connected [14]. Thus when B is connected in Corollary 3.2, there are only three possibilities for the midset.

An example was given in §1 of a nonmanifold midset $M(A, B)$ in E^3 where A is a circle and B is a 2-sphere. Of course A is not convex nor is it simply connected. However B is simply connected, so this condition is not sufficient to insure a manifold midset. We leave to the interested reader the description of arcs A and B in E^3 whose midset is not a manifold.

A classification of all midset manifolds resulting when A is convex would be interesting. Theorem 3.3 gives the situation when A is a singleton set. A natural attempt at such a classification is to prove that a point a can always be chosen in A such that $M(A, B)$ is homeomorphic to $M(\{a\}, B)$, but this is not generally true. In E^2, let $A = \{(x, y) \mid |y| \leqslant 2 \text{ and } x = 0\}$ and let $B = \{(x, y) \mid |x| = 1 \text{ and } |y| \leqslant 1\}$. Then no such point in A exists. Nevertheless, the manifolds listed in Corollary 3.2 seem to be the only ones that are realized in E^3 even when the convex set A is not a singleton set.

4. **Tame midset manifolds.** In the situation where $A = \{a\}$ and $a \notin B$, it is easy to prove the resulting $(n-1)$-manifold $M(A, B)$ tame in E^3 because the radial map h defined in the proof of Theorem 3.1 can be extended to a homeomorphism from a bicollar on $M(A, B)$ to one on the $(n-1)$-sphere $\partial_\varepsilon(A)$. The general situation where neither set is degenerate seems more difficult; nevertheless when midsets are manifolds in E^3 they must be tame.

THEOREM 4.1. *If a 2-manifold K is a subset of the midset M of two disjoint closed subsets A and B of E^3, then K is tame in E^3.*

PROOF. From the hypothesis, M is the common boundary of the disjoint open sets $\{A < B\}$ and $\{B < A\}$ (see [14, Theorem 2]). Let $m \in K$ and let a

be a foot of m on A. The set $\{\{a\} \leqslant B\}$ is convex because it is the intersection of the collection $\{\{\{a\} \leqslant \{b\}\} | b \in B\}$ of closed, convex, half-spaces. Since $\{\{a\} \leqslant B\}$ contains m, is convex, and has a in its interior, it contains a solid right circular cone C_m with vertex m. The cone C_m lies in $\{A \leqslant B\}$ because $\{\{a\} \leqslant B\}$ does. Hence K can be "touched by a pencil" at each $m \in K$ from $\{A < B\}$. Similarly such a cone exists in $\{B \leqslant A\}$.

It follows directly from Corollary 4.6 of [5] that K is tame if K is a 2-sphere. Otherwise, for each $m \in K$, there exist a 2-sphere S and a disk D in $K \cap S$ with $m \in \text{Int } D$ (see [2, Theorem 5.4.1, p. 296]). For convenience we assume D lies in the closure of $(\text{Int } S) \cap \{A < B\}$. Let E be a disk such that $M \in \text{Int } E \subset E \subset \text{Int } D$, and for each $p \in E$ choose a subcone K_p of C_p small enough to lie in $S \cup \text{Int } S$. Let X be the closure of $\bigcup_{p \in E} K_p$, and notice that X is the union of convex 3-cells. Adjust S to obtain a 2-sphere S' such that $X \subset S' \cup \text{Int } S'$ and S' is locally tame modulo X (see [2, Theorem 4.6.1, p. 282]). By Theorem 4.5 of [5], S' is tame from Ext S', which implies that K is locally tame at m from $\{B < A\}$. A similar argument, where the cones K_p are selected in $S \cup \text{Ext } S$, shows K locally tame at m from $\{A < B\}$; hence K is tame.

REMARKS. The above proof generalizes to $E^n (n > 3)$ to show that an $(n-1)$-manifold subset of the midset of disjoint closed sets A and B can be "touched by convex n-cells" from both $\{A < B\}$ and $\{A > B\}$; however, this does not imply tameness for an $(n-1)$-manifold in E^n if $n \geqslant 4$. An example can be obtained using Daverman's "inflation" technique [8, §11].

5. **When an ε-boundary is tame.** Weill [13, p. 248] asked if a 2-sphere S in E^3 is tame when it is, for some $\varepsilon > 0$, the ε-boundary of a compact set X. As he pointed out, the special case where X lies in one component U of $E^3 - S$ can be partially resolved using Theorem 4.5 of [5]; in this case S is tame from $E^3 - (S \cup U)$. However S must be tame from both sides in the case where X intersects both Int S and Ext S; in fact, in this case the result carries over to $E^n (n \neq 4)$.

THEOREM 5.1. *If a compact connected $(n-1)$-manifold M in E^n is the ε-boundary of a set A that intersects both components of $E^n - M$, then M is tame in $E^n (n \neq 4)$.*

PROOF. We may assume A closed since $\partial_\varepsilon(A) = \partial_\varepsilon(\bar{A})$. Define $E = A \cap \text{Ext } M$ and $I = A \cap \text{Int } M$. To show that $\partial_\varepsilon(E) = M$ suppose $p \in \partial_\varepsilon(E) - M$. Since $p \notin \partial_\varepsilon(A)$, we have $d(p, I) < \varepsilon$. But then M fails to separate E from I because there exist segments $[a, p]$ and $[p, e]$ whose union is a path from $a \in I$ to $e \in E$ which fails to intersect M. Thus $\partial_\varepsilon(E) \subset M$, and it follows that $\partial_\varepsilon(E) = M$ since $\partial_\varepsilon(E)$ must separate E^3. Similarly $\partial_\varepsilon(I) = M$.

In case $n = 3$, Theorem 5.1 can be reduced to Theorem 4.1 by showing that M is the midset of I and E; however, we do not make this restriction on n. The strategy is to show that $E^n - M$ is $1 - ULC$, for then the tameness of M follows from [6] or [7] if $n \geqslant 5$ and from Bing's result [2, Theorem 6.4.1] if

$n = 3$. To show $E^n - M$ is $1 - ULC$ we first note that for each point $p \in M$ there are points a and b of I and E, respectively, such that the closed balls $B(a, \varepsilon)$ and $B(b, \varepsilon)$ are tangent at p and lie in the closures of Int M and Ext M, respectively. Griffith [10] proved that such a manifold must be locally spanned from both complementary domains, and Burgess [2, Theorem 6.5.2, p. 305] showed that locally spanned manifolds have $1 - ULC$ complementary domains. Although the Burgess and Griffith results just cited were stated for E^3 the proofs work as well in higher dimensions.

REMARKS. Roughly speaking, Theorem 5.1 says that a manifold M is tame when it is the ε-boundary of a set A which has two pieces separated by a gap of width 2ε. One can extract a seemingly more general theorem to the effect that a manifold ε-boundary $\partial_\varepsilon(A)$ is tame provided it is the midset of A and $\partial_{2\varepsilon}(A)$. However this situation does not seem to arise except in obvious cases (for example when A is convex) in which the tameness of $\partial_\varepsilon(A)$ is clear. In this connection it is interesting to let A be the union of unit intervals [0, 0), (1, 0)] and [(0, 0), (0, 1)] in E^2. There is no positive number ε such that $\partial_\varepsilon(A) = M(A, \partial_{2\varepsilon}(A))$, yet A is a rather simple set.

REFERENCES

1. H. Bell, *Some topological extensions of plane geometry* (manuscript).

2. C. E. Burgess and J. W. Cannon, *Embeddings of surfaces in E^3*, Rocky Mountain J. Math. **1** (1971), 259–344. MR **43** #4008.

3. H. Busemann, *The geometry of geodesics*, Academic Press, New York, 1955. MR **17**, 779.

4. Morton Brown, *Sets of constant distance from a planar set*, Michigan Math. J. **19** (1972), 321–323. MR **47** #4263.

5. J. W. Cannon, **-taming sets for crumpled cubes*. I. *Basic properties*, Trans. Amer. Math. Soc. **161** (1971), 429–440. MR **43** #8065.

6. A. V. Černavskiĭ, *Coincidence of local flatness and local simple-connectedness for embeddings of $(n-1)$-dimensional manifolds in n-dimensional manifolds when $n > 4$*, Mat. Sb. (N.S.) **91** **(133)** (1973), 279–286, 288 = Math. USSR Sbornik **20** (1973), 297–304. MR **48** #12541.

7. R. J. Daverman, *Locally nice codimension one manifolds are locally flat*, Bull. Amer. Math. Soc. **79** (1973), 410–413. MR **47** #9628.

8. ———, *Sewings of closed n-cell-complements*, Trans. Amer. Math. Soc. (to appear).

9. Steve Ferry, *When ε-boundaries are manifolds*, Fund. Math. (to appear).

10. H. C. Griffith, *Spheres uniformly wedged between balls are tame in E^3*, Amer. Math. Monthly **75** (1968), 767. MR **38** #2753.

11. R. Gariepy and W. D. Pepe, *On the level sets of a distance function in a Minkowski space*, Proc. Amer. Math. Soc. **31** (1972), 255–259. MR **44** #4646.

12. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Univ. Press, Princeton, N.J., 1941. MR **3**, 312.

13. L. R. Weill, *A new characterization of tame 2-spheres in E^3*, Trans. Amer. Math. Soc. **190** (1974), 243–252. MR **49** #3951.

14. J. B. Wilker, *Equidistant sets and their connectivity properties*, Proc. Amer. Math. Soc. **47** (1975), 446–452. MR **50** #8265.

DEPARTMENT OF MATHEMATICS, UTAH STATE UNIVERSITY, LOGAN, UTAH 84321