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FIXED POINTS OF ANOSOV MAPS OF CERTAIN MANIFOLDS

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ABSTRACT. LEMMA. *If H is a graded exterior algebra on odd generators with augmentation ideal J and h: $H \to H$ is an algebra homomorphism inducing $J/J^2 \to J/J^2$ with eigenvalues $\{\lambda_i\}$, then the Lefschetz number* $L(h) = \prod(1 - \lambda_i)$. The lemma is applied to an Anosov map or diffeomorphism of a compact manifold with real cohomology H to give sufficient conditions that none of the eigenvalues λ_i be a root of unity and that there exist a fixed point. In particular, every Anosov diffeomorphism of a compact connected Lie group has a fixed point.

1. **Introduction and statement of results.** An *Anosov map* of a smooth manifold M is a smooth map f: $M \to M$ such that (1) There is a splitting of the tangent bundle $T(M)$ into a continuous Whitney sum $T(M) = E^s + E^u$ of subbundles which is invariant under the derivative map Df: $T(M) \to T(M)$.

(2) There exists a Riemannian metric $\| \ \|$ on $T(M)$ and constants $C > 0$, $C' > 0$, $0 < \lambda < 1$, such that

$$\|Df^m(v)\| \leqslant C\lambda^m\|v\| \quad \text{and} \quad \|DF^m(w)\| \geqslant C'\lambda^{-m}\|w\|$$

for all $v \in E^s$, $w \in E^u$, and $m \in Z^+$. One checks easily (see [9, §3.1]) that for M compact the second condition is independent of which Riemannian metric is chosen.

An *Anosov diffeomorphism* is an Anosov map which is a diffeomorphism. Examples on the n-torus $T^n = S^1 \times \ldots \times S^1$ are gotten by taking a matrix $f_0 \in GL(n, Z)$ none of whose eigenvalues has absolute value 1. Then f_0 induces an automorphism f of $T^n = R^n/Z^n$ which, it is not hard to show, is an Anosov diffeomorphism. Nontoral examples have been given by Smale [9, §1.3] on nilmanifolds and by Shub [8, p. 189] on infranilmanifolds.

Examples of Anosov maps g can be constructed on products $M \times N \times P$ by taking $g = p \circ (\mathrm{id}_M \times f \times e)$, where p: $M \times N \times P \to * \times N \times P \subset M \times N \times P$ is projection, f: $N \to N$ is an Anosov diffeomorphism, and e: $P \to P$ is an expanding map (= Anosov map with $E^s = 0$; see Shub [8]).

In [9, § 3.4], Smale asks whether every Anosov diffeomorphism has a fixed

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point. We obtain the following partial answers. Say that an algebra H over R is a *T-algebra* if H is isomorphic to the real cohomology algebra of a product of odd-dimensional spheres; i.e., H is a graded exterior algebra on generators of odd degree. A manifold whose real cohomology is a T-algebra will be called a *T-manifold.*

THEOREM 1. *Let f be an Anosov map of a compact T-manifold M. If E^u is orientable and f has a periodic point then f has a fixed point.*

THEOREM 2. *If M is a compact T-manifold, then every Anosov diffeomorphism of M with E^s or E^u orientable has a fixed point.*

COROLLARY. *Every Anosov diffeomorphism of a compact, connected Lie group G or of G/F, F a finite subgroup of G, has a fixed point.*

This generalizes the case $G = T^n$ proved by Franks [3] and Manning [7]. The following result, which restricts the homotopy classes that admit Anosov diffeomorphisms, is also an extension of work of Franks [3] on tori and Manning [6], [7] on infranilmanifolds. See also Hirsch [4].

THEOREM 3. *Let f be an Anosov diffeomorphism of a compact T-manifold M, with E^u or E^s orientable. Denote by J the augmentation ideal of $H^*(M; R)$ consisting of positive dimensional elements. Then no eigenvalue of $\bar{f}^*: J/J^2 \to J/J^2$ is a root of unity.*

The main tool used in the proofs of these results is the following algebraic lemma, which may be of independent interest.

LEMMA. *Let H be a T-algebra with augmentation ideal J and let $h: H \to H$ be an algebra homomorphism inducing $J/J^2 \to J/J^2$ with eigenvalues $\{\lambda_i\}$. Then the Lefschetz number $L(h^q) = \prod_i(1 - \lambda_i^q)$ for $q > 0$.*

Recall the definition

$$L(h) = \sum_{k=0}^{n} (-1)^k \operatorname{trace}(h|H_k: H_k \to H_k),$$

where $H = H_0 \oplus H_1 \oplus \ldots \oplus H_n$ is the grading of H.

I do not know whether any of the compact manifolds other than tori which admit Anosov diffeomorphisms are T-manifolds. Some of the nilmanifold examples definitely are not T-manifolds.

Finally, I would like to thank John Milnor and the referee for reformulating Theorem 3 and the Lemma and shortening their proofs.

2. **Proofs.**

PROOF OF LEMMA. h induces an algebra homomorphism from the direct sum $H/J \oplus J/J^2 \oplus J^2/J^3 \oplus \ldots$ to itself. This induced homomorphism has the same Lefschetz number. But J^r/J^{r+1} can be identified with the rth exterior power $E^r(J/J^2)$. So if $\bar{h}$ denotes the induced map $J/J^2 \to J/J^2$, then $L(h) = \sum L(E^r\bar{h}) = \sum(-1)^r$ trace $E^r\bar{h}$ since $E^r(J/J^2)$ is concentrated in even or odd dimensions according as r is even or odd. But trace $E^r\bar{h}$ equals the rth elementary symmetric function of the eigenvalues λ_i. (This is proved by putting $\bar{h}$ in triangular form over a suitable extension field.) Hence $L(h) = \Pi(1 - \lambda_i)$. The Lemma follows.

REMARK. The proof works over any field.

PROOF OF THEOREM 1. Assume f has no fixed points. Then $L(f^*) = 0$ since E^u orientable implies $|L(f^*)| = \text{Card}(\text{Fix}(f))$, see [3, p. 123]. Then some $\lambda_i = 1$, using the Lemma with $H = H^*(M; R)$ and $h = f^*$. Hence

$$\text{Card}(\text{Fix}(f^q)) = |L(f^q)^*| = 0 \quad \text{for all } q > 0,$$

again using the Lemma, so f can have no periodic points. This contradicts the hypothesis and proves the theorem.

PROOF OF THEOREM 2. Either E^s or E^u is orientable. We may assume it is E^u, since otherwise we may replace f by the Anosov diffeomorphism f^{-1}. Theorem 2 now follows from Theorem 1 using the fact that every Anosov diffeomorphism has a periodic point (Proposition 1.7 of [2]).

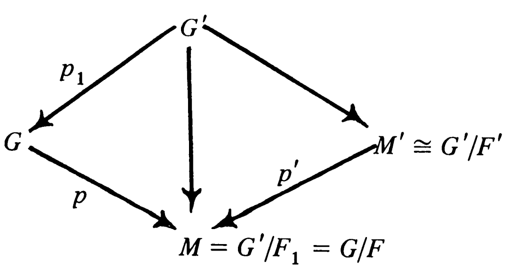
PROOF OF COROLLARY. We are given an Anosov diffeomorphism f of a quotient $M = G/F$, where F is a finite subgroup of a compact, connected Lie group G.

Case I. E^u is orientable. By a theorem of Hopf (see [5] or [1, Chapter I]), G is a T-manifold. A well-known easy argument shows that since G is connected, $H^*(G; R) \cong H^*(M; R)$. Thus M is a T-manifold and Theorem 2 implies f has a fixed point.

Case II. E^u is nonorientable. Then f lifts to an Anosov diffeomorphism f': $M' \to M'$ of a connected 2-fold covering M' of M with $E^{u'}$ orientable. The following result implies, as above, that M' is a T-manifold. Hence f', and therefore f, has a fixed point, completing the proof of the Corollary.

Claim. There exists a compact, connected Lie group G' and a finite subgroup F' such that G'/F' is diffeomorphic to M'.

PROOF. Let p: $G \to G/F = M$ and p': $M' \to M$ be the projection maps. Set $\pi = p_*\pi_1 G \cap p'_*\pi_1 M'$ and let p_0: $G' \to M$ be the covering space such that $p_{0*}\pi_1 G' = \pi$. Now G' is also the covering space of G associated to the subgroup $p_*^{-1}(\pi)$ of $\pi_1 G$.



It is well known that G' is therefore a connected Lie group and the covering projection p_1: $G' \to G$ is a homomorphism. Clearly $[\pi_1 M: \pi] < \infty$, so G' is compact.

Let F_1 denote the subgroup $p_1^{-1}(F)$ of G'. There is a natural isomorphism $h: \pi_1 M/\pi \to F_1$. Set $F' = h(p'_* \pi_1 M'/\pi) \subset F_1$. Then F' is a finite subgroup of G' and the natural projection $G'/F' \to G'/F_1 = M$ induces a monomorphism of fundamental groups which carries $\pi_1(G'/F')$ onto $p'_* \pi_1 M'$. It follows from the uniqueness of covering spaces associated to a subgroup that G'/F' is diffeomorphic to M'. This completes the proof.

PROOF OF THEOREM 3. Suppose some eigenvalue of $\bar{f}^*$ is a qth root of unity, $q > 0$. As in the proof of Theorem 2, we may assume E^u is orientable. Then $\operatorname{Card}(\operatorname{Fix}(f^q)) = |L(f^q)^*| = 0$ by the Lemma, contradicting Theorem 2.

REMARK. A 2-fold cover of a T-manifold might not be a T-manifold, e.g., $S^1 \times RP^2$. Thus the hypothesis of orientability in the theorems cannot be dispensed with as in the proof of the Corollary.

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