FIXED POINTS OF ANOSOV MAPS OF CERTAIN MANIFOLDS

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ABSTRACT. Lemma. If $H$ is a graded exterior algebra on odd generators with augmentation ideal $J$ and $h: H \to H$ is an algebra homomorphism inducing $J/J^2 \to J/J^2$ with eigenvalues $\{\lambda_i\}$, then the Lefschetz number $L(h) = (1 - \lambda_1)$. The lemma is applied to an Anosov map or diffeomorphism of a compact manifold with real cohomology $H$ to give sufficient conditions that none of the eigenvalues $\lambda_i$ be a root of unity and that there exist a fixed point. In particular, every Anosov diffeomorphism of a compact connected Lie group has a fixed point.

1. Introduction and statement of results. An Anosov map of a smooth manifold $M$ is a smooth map $f: M \to M$ such that (1) There is a splitting of the tangent bundle $T(M)$ into a continuous Whitney sum $T(M) = E^s + E^u$ of subbundles which is invariant under the derivative map $Df: T(M) \to T(M)$.

(2) There exists a Riemannian metric $\|\|$ on $T(M)$ and constants $C > 0$, $C' > 0$, $0 < \lambda < 1$, such that

$$\|Df^m(v)\| \leq C\lambda^m\|v\|$$

and

$$\|DF^m(w)\| \geq C'\lambda^{-m}\|w\|$$

for all $v \in E^s$, $w \in E^u$, and $m \in \mathbb{Z}^+$. One checks easily (see [9, §3.1]) that for $M$ compact the second condition is independent of which Riemannian metric is chosen.

An Anosov diffeomorphism is an Anosov map which is a diffeomorphism. Examples on the $n$-torus $T^n = S^1 \times \ldots \times S^1$ are gotten by taking a matrix $f_0 \in GL(n, \mathbb{Z})$ none of whose eigenvalues has absolute value 1. Then $f_0$ induces an automorphism $f$ of $T^n = R^n/Z^n$ which, it is not hard to show, is an Anosov diffeomorphism. Nontoral examples have been given by Smale [9, §1.3] on nilmanifolds and by Shub [8, p. 189] on infranilmanifolds.

Examples of Anosov maps $g$ can be constructed on products $M \times N \times P$ by taking $g = p \circ (id_M \times f \times e)$, where $p: M \times N \times P \to * \times N \times P \subseteq M \times N \times P$ is projection, $f: N \to N$ is an Anosov diffeomorphism, and $e: P \to P$ is an expanding map (= Anosov map with $E^s = 0$; see Shub [8]).

In [9, §3.4], Smale asks whether every Anosov diffeomorphism has a fixed point.
point. We obtain the following partial answers. Say that an algebra $H$ over $R$ is a $T$-algebra if $H$ is isomorphic to the real cohomology algebra of a product of odd-dimensional spheres; i.e., $H$ is a graded exterior algebra on generators of odd degree. A manifold whose real cohomology is a $T$-algebra will be called a $T$-manifold.

**Theorem 1.** Let $f$ be an Anosov map of a compact $T$-manifold $M$. If $E^n$ is orientable and $f$ has a periodic point then $f$ has a fixed point.

**Theorem 2.** If $M$ is a compact $T$-manifold, then every Anosov diffeomorphism of $M$ with $E^3$ or $E^n$ orientable has a fixed point.

**Corollary.** Every Anosov diffeomorphism of a compact, connected Lie group $G$ or of $G/F$, $F$ a finite subgroup of $G$, has a fixed point.

This generalizes the case $G = T^n$ proved by Franks [3] and Manning [7]. The following result, which restricts the homotopy classes that admit Anosov diffeomorphisms, is also an extension of work of Franks [3] on tori and Manning [6], [7] on infranilmanifolds. See also Hirsch [4].

**Theorem 3.** Let $f$ be an Anosov diffeomorphism of a compact $T$-manifold $M$, with $E^n$ or $E^3$ orientable. Denote by $J$ the augmentation ideal of $H^*(M; R)$ consisting of positive dimensional elements. Then no eigenvalue of $f^*: J/J^2 \to J/J^2$ is a root of unity.

The main tool used in the proofs of these results is the following algebraic lemma, which may be of independent interest.

**Lemma.** Let $H$ be a $T$-algebra with augmentation ideal $J$ and let $h: H \to H$ be an algebra homomorphism inducing $J/J^2 \to J/J^2$ with eigenvalues $\{\lambda_i\}$. Then the Lefschetz number $L(h^q) = \prod_i (1 - \lambda_i^q)$ for $q > 0$.

Recall the definition

$$L(h) = \sum_{k=0}^{n} (-1)^k \text{trace}(h|H_k: H_k \to H_k),$$

where $H = H_0 \oplus H_1 \oplus \ldots \oplus H_n$ is the grading of $H$.

I do not know whether any of the compact manifolds other than tori which admit Anosov diffeomorphisms are $T$-manifolds. Some of the nilmanifold examples definitely are not $T$-manifolds.

Finally, I would like to thank John Milnor and the referee for reformulating Theorem 3 and the Lemma and shortening their proofs.

2. **Proofs.**

**Proof of Lemma.** $h$ induces an algebra homomorphism from the direct sum $H/J \oplus J/J^2 \oplus J^2/J^3 \oplus \ldots$ to itself. This induced homomorphism has the same Lefschetz number. But $J'/J'^{r+1}$ can be identified with the $r$th exterior power $E^r(J/J^2)$. So if $\tilde{h}$ denotes the induced map $J/J^2 \to J/J^2$, then $L(h) = \sum L(E'h) = \sum (-1)^r \text{trace} E'h$ since $E^r(J/J^2)$ is concentrated...
in even or odd dimensions according as \( r \) is even or odd. But trace \( E^r \) equals the \( r \)th elementary symmetric function of the eigenvalues \( \lambda_i \). (This is proved by putting \( h \) in triangular form over a suitable extension field.) Hence \( L(h) = \prod(1 - \lambda_i) \). The Lemma follows.

Remark. The proof works over any field.

Proof of Theorem 1. Assume \( f \) has no fixed points. Then \( L(f^*) = 0 \) since \( E^u \) orientable implies \( |L(f^*)| = \text{Card}(\text{Fix}(f)) \), see [3, p. 123]. Then some \( \lambda_i = 1 \), using the Lemma with \( H = H^*(M; R) \) and \( h = f^* \). Hence

\[
\text{Card}(\text{Fix}(f^q)) = |L(f^q^*)| = 0 \quad \text{for all } q > 0,
\]
again using the Lemma, so \( f \) can have no periodic points. This contradicts the hypothesis and proves the theorem.

Proof of Theorem 2. Either \( E^s \) or \( E^u \) is orientable. We may assume it is \( E^u \), since otherwise we may replace \( f \) by the Anosov diffeomorphism \( f^{-1} \). Theorem 2 now follows from Theorem 1 using the fact that every Anosov diffeomorphism has a periodic point (Proposition 1.7 of [2]).

Proof of Corollary. We are given an Anosov diffeomorphism \( f \) of a quotient \( M = G/F \), where \( F \) is a finite subgroup of a compact, connected Lie group \( G \).

Case I. \( E^u \) is orientable. By a theorem of Hopf (see [5] or [1, Chapter I]), \( G \) is a \( T \)-manifold. A well-known easy argument shows that since \( G \) is connected, \( H^*(G; R) \cong H^*(M; R) \). Thus \( M \) is a \( T \)-manifold and Theorem 2 implies \( f \) has a fixed point.

Case II. \( E^u \) is nonorientable. Then \( f \) lifts to an Anosov diffeomorphism \( f': M' \to M' \) of a connected 2-fold covering \( M' \) of \( M \) with \( E^u \) orientable. The following result implies, as above, that \( M' \) is a \( T \)-manifold. Hence \( f' \), and therefore \( f \), has a fixed point, completing the proof of the Corollary.

Claim. There exists a compact, connected Lie group \( G' \) and a finite subgroup \( F' \) such that \( G'/F' \) is diffeomorphic to \( M' \).

Proof. Let \( p: G \to G/F = M \) and \( p': M' \to M \) be the projection maps. Set \( \pi = p_\ast \pi_1 G \cap p'_\ast \pi_1 M' \) and let \( p_0: G' \to M \) be the covering space such that \( p_0\ast \pi_1 G' = \pi \). Now \( G' \) is also the covering space of \( G \) associated to the subgroup \( p_\ast \pi_1^{-1}(\pi) \) of \( \pi_1 G \).

It is well known that \( G' \) is therefore a connected Lie group and the covering projection \( p_1: G' \to G \) is a homomorphism. Clearly \( [\pi_1 M; \pi] < \infty \), so \( G' \) is compact.
Let $F_1$ denote the subgroup $p_1^{-1}(F)$ of $G'$. There is a natural isomorphism $h: \pi_1 M'/\pi \to F_1$. Set $F' = h(p'_*\pi_1 M'/\pi) \subset F_1$. Then $F'$ is a finite subgroup of $G'$ and the natural projection $G'/F' \to G'/F_1 = M$ induces a monomorphism of fundamental groups which carries $\pi_1(G'/F')$ onto $p'_*\pi_1 M'$. It follows from the uniqueness of covering spaces associated to a subgroup that $G'/F'$ is diffeomorphic to $M'$. This completes the proof.

**Proof of Theorem 3.** Suppose some eigenvalue of $\tilde{f}^*$ is a $q$th root of unity, $q > 0$. As in the proof of Theorem 2, we may assume $E''$ is orientable. Then $\text{Card}(\text{Fix}(f^q)) = |L(f^q)| = 0$ by the Lemma, contradicting Theorem 2.

**Remark.** A 2-fold cover of a $T$-manifold might not be a $T$-manifold, e.g., $S^1 \times RP^2$. Thus the hypothesis of orientability in the theorems cannot be dispensed with as in the proof of the Corollary.

**References**


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