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ON THE L^2-BOUNDEDNESS OF PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. A. P. Calderón and R. Vaillancourt have established L^2-boundedness in case the symbol has bounded partial derivatives to order 3 in any variable. H. O. Cordes has proved L^2-boundedness if this order is 1. It is shown here that it suffices for the symbol to satisfy a uniform Hölder continuity condition of order $\frac{1}{2} + \delta$, $\delta > 0$.

To state our result for n dimensions, it is useful to define a difference operator similar to the standard differential operator notation.

DEFINITION. Let a be a complex-valued function of N real variables. Then, the shift operator is defined by

$$(S_h^\beta a)(X_1, \dots, X_N) \triangleq a(X_1 + \beta_1 h_1, \dots, X_N + \beta_N h_N)$$

where $h = (h_1, \dots, h_N) \in R^N$ and $\beta = (\beta_1, \dots, \beta_N)$ is a multi-index. The entries of β are restricted to 0 and 1, and, for the shift operator, $|\beta| \triangleq \beta_1 + \cdots + \beta_N = 1$. The difference operator is defined by

$$D_h^\beta \triangleq \begin{cases} I, \text{ the identity operator} & \text{for } |\beta| = 0, \\ S_h^\beta - I & \text{for } |\beta| = 1, \\ D_h^{\beta_1} \cdots D_h^{\beta_N} & \text{for } |\beta| \geqslant 2, \end{cases}$$

where β^i is a multi-index with all its entries equal to 0 except the ith, which is equal to β_i.

In this notation, our main result is

THEOREM. *Let a be a complex-valued function of $N = 2n$ real variables. Suppose that for all $(X_1, \dots, X_N) \in R^N$ and for sufficiently small $h \in R^N$, there exist nonnegative constants $\{c_\beta\}$ and there exists $\delta > 0$ such that*

$$|D_h^\beta a(X_1, \dots, X_N)| \leqslant c_\beta |h_1|^{\beta_1(1/2+\delta)} \cdots |h_N|^{\beta_N(1/2+\delta)} \tag{1}$$

for all β's with entries restricted to 0 *or* 1. *Let the transformation A be defined by*

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$$(Au)(x) \triangleq (2\pi)^{-n/2} \int_{R^n} e^{i(x\cdot z)} a(x,z)\hat{u}(z)\,dz$$

for all $u \in \mathbb{S}(R^n)$, *the Schwartz class, where* $\hat{}$ *is the Fourier transform. Then* A *can be uniquely extended to be a bounded linear operator on* $L^2(R^n)$.

PROOF. The method used will be that of Cordes [2]. Cordes proves his result for a smoothness condition on the symbol which is expressed in terms of differentiation with respect to x and z instead of with respect to the separate components x_i and z_i, $i = 1, \ldots, n$ (as in Calderón and Vaillancourt [1] and in this paper). Cordes shows that if

$$b(x,z) \triangleq (1-\Delta_x)^{1/4+\varepsilon}(1-\Delta_z)^{1/4+\varepsilon} a(x,z)$$

is in $L^\infty(R^N)$ for any $\varepsilon > 0$, then A is L^2-bounded. We will prove

$$b(X_1,\ldots,X_N) = (1-\partial^2/\partial X_1^2)^{1/4+\varepsilon}\cdots(1-\partial^2/\partial X_N^2)^{1/4+\varepsilon} a(X_1,\ldots,X_N)$$

is in $L^\infty(R^N)$ for $0 < \varepsilon < \delta/2$ and $N = 1, 2, \ldots$. This suffices for the L^2-boundedness of A. The modification of Cordes' proof is completely straightforward and obvious. The details are given in [3].

The proof is by induction on N. Both the case $N = 1$ and the induction step from $N = k$ to $N = k+1$ follow readily from the

LEMMA. *If* a *satisfies hypothesis* (1) *with* $N = 1$, *then*

$$b(x) \triangleq \left(1 - \frac{d^2}{dx^2}\right)^{1/4+\varepsilon} a(x) = a(x) + \int_R \psi_{-1/4-\varepsilon}(x-y)(a(y)-a(x))\,dy$$

where

$$\psi_s(x) \triangleq (2\pi)^{-1/2}(2^{1-s}/\Gamma(s))|x|^{s-1/2} K_{s-1/2}(|x|)$$

where Γ *is the gamma function and* K_σ *is the modified Hankel function of order* σ.

For $N = 1$, that b is in L^∞ results easily from hypothesis (1) and properties 1 and 2 of $\psi_s(x)$ listed below. For the induction step, a general expression for $b(X_1,\ldots,X_N)$ is demonstrated (it is here that we use the Lemma to go from $N = k$ to $N = k+1$). This expression, which is easily shown to be bounded, is

$$b(X_1,\ldots,X_N) = a(X_1,\ldots,X_N) + \sum_{|\beta|=1}^{N} \int_{R^{|\beta|}} D^\beta_{W-X} a(X_1,\ldots,X_N) \prod_{j=1}^{|\beta|} \psi_{-1/4-\varepsilon}(X_{i_j} - W_{i_j})\,dW_{i_j}$$

where $W - X = (W_1 - X_1, \ldots, W_N - X_N)$; $i_1, \ldots, i_{|\beta|}$ are the indices corresponding to nonzero entries of β. Again, the details are given in [3].

Proof of the Lemma. We use these properties of $\psi_s(x)$:

1. $\psi_s(x)$ decays exponentially as $|x|$ goes to ∞;
2. $\psi_s(x)$ is asymptotically like $|x|^{2\mathrm{Re}(s)-1}$ as $|x|$ goes to zero;
3. $\psi_s(x)$ is analytic in s and C^∞ in (s, x) away from $|x| = 0$ in the strip $-\frac{1}{2} < \mathrm{Re}(s) < \frac{1}{2}$;
4. $(2\pi)^{1/2}\hat{\psi}_s(x) = (1 + x^2)^{-s}$ $(0 < \mathrm{Re}(s) < \frac{1}{2})$.

Cordes, already, mentions all of these in [2]. It suffices by the standard duality to consider the case $a = u \in \mathcal{S}$, the Schwartz class of rapidly decreasing functions. By 4, if $0 < \mathrm{Re}(s) < \frac{1}{2}$,

$$
\begin{aligned}
(1 - d^2/dx^2)^{-s}u(x) &= (F^{-1}(1 + x^2)^{-s}Fu)(x) = F^{-1}((2\pi)^{1/2}\hat{\psi}_s(x)\hat{u}(x)) \\
&= (\psi_s * u)(x) = \int_R \psi_s(x - y)u(y)\,dy \\
&= u(x)\int_R e^{-i0y}\psi_s(y)dy + \int_R \psi_s(x - y)(u(y) - u(x))\,dy \qquad (2) \\
&= u(x)(2\pi)^{1/2}\hat{\psi}_s(0) + \int_R \psi_s(x - y)(u(y) - u(x))\,dy \\
&= u(x) + \int_R \psi_s(x - y)(u(y) - u(x))\,dy,
\end{aligned}
$$

where F is the Fourier transform. But, by hypothesis (1), $|u(y) - u(x)| \leqslant c_1|y - x|^{1/2+\delta}$ for sufficiently small $|y - x|$. So, using 1 and 2, the integral on the right is absolutely convergent if $\mathrm{Re}(s) > -\frac{1}{4} - \varepsilon$, for then $2\,\mathrm{Re}(s) - 1 + \frac{1}{2} + \delta > -1$ since $\varepsilon < \delta/2$. The Lemma follows from uniqueness of analytic continuation if we can show both sides of equation (2) are analytic in the strip $-\frac{1}{4} - \varepsilon < \mathrm{Re}(s) < \frac{1}{2}$ (with x and u fixed). But,

$$\left(1 - \frac{d^2}{dx^2}\right)^{-s} u(x) = (2\pi)^{-1/2}\int_R e^{-ixz}(1 + z^2)^{-s}\hat{u}(z)\,dz$$

is easily seen to be entire by Weierstrass' Theorem. And, using properties 1, 2, and especially 3, it is likewise easy to establish (via the Cauchy-Riemann conditions and the Weierstrass Theorem) that $\int_R \psi_s(x - y)(u(y) - u(x))\,dy$ is analytic on the strip. Q.E.D. Lemma and Theorem.

References

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