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ON ASYMPTOTIC VALUES OF ANALYTIC FUNCTIONS ON RIEMANN SURFACES

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ABSTRACT. Some generalizations of Lindelöf's theorems on asymptotic values of bounded analytic functions are given on subregions of Riemann surfaces.

Let R be an open Riemann surface. Let R^* denote a metrizable compactification of R, and put $\Delta = R^* - R$. $\bar{A}$ means the closure of a set $A \subset R^*$ with respect to R^*. ∂A means the relative boundary of $A \subset R$ with respect to R. Let G be a region, which is not relatively compact on R, with the property that ∂G consists of a finite number of noncompact Jordan arcs C_n ($n = 1, 2, \ldots, N$), and that $\bar{G} \cap \Delta$ is a single point p.

Each point q of ∂G is accessible in G. It is said that a Jordan arc J : $a = g(t)$ $(0 \leqslant t < 1)$ decides an accessible boundary point $q(J)$ in G, when $J \subset G$ and $\lim_{t \to 1} g(t) = q$. Let Jordan arcs J_1 and J_2 decide accessible boundary points $q(J_1)$ and $q(J_2)$ in G, respectively. Let $V(q)$ be any parametric disk about q satisfying $J_1 \cap \partial V(q) \neq \varnothing$ and $J_2 \cap \partial V(q) \neq \varnothing$. Let J_1' and J_2' denote, respectively, the components of $J_1 \cap V(q)$ and $J_2 \cap V(q)$ which are not relatively compact on G. We say that $q(J_1)$ and $q(J_2)$ are identical when two points $q_1 \in \bar{J_1'} \cap \partial V(q)$ and $q_2 \in \bar{J_2'} \cap \partial V(q)$ can be joined by a Jordan arc $J^* \subset G \cap \overline{V(q)}$. If not, then it is said that $q(J_1)$ and $q(J_2)$ are distinct.

In this sense, let each point of ∂G be distinguished, and let $h_n(t)$ $(0 \leqslant t < 1)$ denote a parametric representation of C_n.

Let h be any bounded continuous real-valued function on $\partial G \cup \{p\}$. Since $h - h(p)$ is resolutive (cf. [1, Theorem 3.2]), h is resolutive (cf. [1, Theorem 8.1]). Therefore $G^* = G \cup \partial G \cup \{p\}$ is a resolutive compactification of G with respect to the relative topology of G^* for R^* (cf. [1, p. 87]).

Henceforth we assume that p is regular with respect to G^* in the sense of the Dirichlet problem, and that $\{p\}$ is of harmonic measure 0 with respect to G^*.

In this paper we shall show the following Theorem and its applications.

THEOREM. *Let f be a bounded holomorphic function on G which is continuous*

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on $G \cup \partial G$. *If* $\limsup_{C_n \ni a \to p} |f(a)| \leqslant m$ *for each* n, *then* $\limsup_{G \ni a \to p} |f(a)| \leqslant m$.

PROOF. We choose an M such that $|f| \leqslant M$ on G and $m < M$. For any $\varepsilon > 0$ ($\varepsilon < M - m$) and each n, there exists a T_n such that $|f| \leqslant m + \varepsilon$ on $\{h_n(t); t \geqslant T_n\}$. Let $U_0(p)$ be a neighborhood of p satisfying $\partial G \cap U_0(p) \subset \bigcup_n h_n(t)$ for all $t \geqslant T_n$. Let h_0 be a nonnegative continuous function on $\partial G \cup \{p\}$ which is equal to 0 at p and is equal to $\log M/(m + \varepsilon)$ on $\partial G \cap (R - U_0(p))$.

Let u be the solution of the Dirichlet problem on G with h_0 as a boundary function, and let v be a conjugate harmonic function of u on G. We put $F = fe^{-u-iv}$. Since $e^{-u} \leqslant 1$ on $G \cup \partial G$, we have $|F| \leqslant M$ on G and $|F| \leqslant m + \varepsilon$ on ∂G.

$\{p\}$ is polar with respect to G^* (cf. [1, p. 94]). Let s be a positive superharmonic function on G with $\lim_{G \ni a \to p} s(a) = \infty$; then for any $\varepsilon' > 0$,

$$\liminf_{G \ni a \to \partial G \cup \{p\}} \big(-|F(a)| + m + \varepsilon + \varepsilon' s(a)\big) \geqslant 0.$$

Therefore from the minimal principle (cf. [1, Theorem 1.2]), we have $|F| \leqslant m + \varepsilon + \varepsilon' s$ on G and, hence, $|F| \leqslant m + \varepsilon$ on it.

Since $\lim_{G \ni a \to p} e^u = 1$, there exists a neighborhood $U^*(p)$ satisfying $e^u \leqslant 1 + \varepsilon$ on $G \cap U^*(p)$. Thus $|f| \leqslant (m + \varepsilon)(1 + \varepsilon)$ on $G \cap U^*(p)$ and, hence, $\limsup_{G \ni a \to p} |f(a)| \leqslant m$.

By applying the Theorem to $f - c$, we have

COROLLARY 1. *For f as defined in the Theorem, if* $\lim_{C_n \ni a \to p} f(a) = c$ *for each n, then* $\lim_{G \ni a \to p} f(a) = c$.

Henceforth let $\lim_{C_n \ni a \to p} f(a) = c'$ for each n ($n = 1, 2, \ldots, m$), and let $\lim_{C_n \ni a \to p} f(a) = c''$ for each n ($n = m + 1, m + 2, \ldots, N$), where c' and c'' are finite.

COROLLARY 2. *If f is a function defined as in the Theorem, then $c' = c''$ and* $\lim_{G \ni a \to p} f(a) = c'$.

PROOF. We put $H = (f - c')(f - c'')$ and get $\lim_{C_n \ni a \to p} H(a) = 0$. For any $\varepsilon > 0$, there exist t_1 and t_2 such that $|f - c'| \leqslant \sqrt{\varepsilon}$ on $\{h_1(t); t \geqslant t_1\}$ and $|f - c''| \leqslant \sqrt{\varepsilon}$ on $\{h_{m+1}(t); t \geqslant t_2\}$. From Corollary 1, there exists a neighborhood $U'(p)$ such that $\{h_1(t); t \geqslant t_1\} \cap (R - U'(p)) \neq \varnothing$, $\{h_{m+1}(t); t \geqslant t_2\} \cap (R - U'(p)) \neq \varnothing$ and $|H| \leqslant \varepsilon$ on $G \cap U'(p)$.

Let D be the component of $G \cap U'(p)$ which is not relatively compact; then a point a_1 of $\{h_1(t); t \geqslant t_1\} \cap D$ and a point a_2 of $\{h_{m+1}(t); t \geqslant t_2\} \cap D$ can be joined by a Jordan arc $C_0 \subset D$. If $|f(a_2) - c'| \leqslant \sqrt{\varepsilon}$, then $|c' - c''| \leqslant 2\sqrt{\varepsilon}$. If $|f(a_2) - c'| > \sqrt{\varepsilon}$, then there exists a point $a_0 \in C_0$ satisfying $|f(a_0) - c'| = \sqrt{\varepsilon}$. Thus $|f(a_0) - c''| \leqslant \sqrt{\varepsilon}$ and, hence, $c' = c''$.

From Corollary 2 and the proof of Lindelöf's theorem (cf. [2, pp. 307–308]) we have

COROLLARY 3. *Let f be meromorphic on $G \cup \partial G$. If f omits three distinct values c_1, c_2 and c_3 in G, then $c' = c''$ and* $\lim_{G \ni a \to p} f(a) = c'$.

From Corollary 3 we have

COROLLARY 4. *If f is meromorphic on $G \cup \partial G$ and $c' \neq c''$, then every value of $\{|w| \leqslant \infty\}$ is assumed in G infinitely often by f with at most two exceptions.*

COROLLARY 5. *Let f be a holomorphic function on G which is continuous on $G \cup \partial G$. If $c' \neq c''$, then there exists an asymptotic path L converging to p with* $\lim_{L \ni a \to p} f(a) = \infty$.

PROOF. From Corollary 2, f is unbounded on G. Let $U_n(p)$ denote the $1/n$-neighborhood of p, and let N^* be an integer satisfying $N^* > \sup_{a \in \partial G} |f(a)|$. Let D_n be a component, which is not relatively compact on $G \cup \partial G$, of $\{a \in G; |f(a)| > n\} \cap U_n(p)$ for each $n \geqslant N^*$ such that $D_{n+1} \subset D_n$. We take a point $a_n \in D_n$ for each $n \geqslant N^*$, and join a_n and a_{n+1} by a Jordan arc $L_n \subset D_n$ such that $L_{n+1} \cap (\bigcup_{k=N^*}^{n} L_k - \{a_{n+1}\}) = \emptyset$. $L = \bigcup_n L_n$ converges to p and $\lim_{L \ni a \to p} f(a) = \infty$.

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