SHORTER NOTES

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A NOTE ON THE DENJOY INTEGRABILITY OF ABSTRACTLY-VALUED FUNCTIONS

T. J. MORRISON

Abstract. We present necessary and sufficient conditions on a Banach space $X$ that the classes of strongly measurable $X$-valued Denjoy-Pettis integrable and Denjoy-Gelfand integrable functions coincide.

In this note we give necessary and sufficient conditions on a Banach space $X$ that the classes of strongly measurable, $X$-valued Denjoy-Pettis and Denjoy-Gelfand integrable functions coincide. We preserve here, for the most part, the notations of D. W. Solomon [3] and, for the sake of simplicity, restrict our attention to Lebesgue measure $\lambda$ on $[0, 1]$.

Throughout, following Solomon, we let $DP$ denote the Denjoy-Gelfand integral and $DP^*$ the Denjoy-Pettis integral. All point functions $f$ will be strongly measurable, defined on $[0, 1]$ and take values in the Banach space $X$. Letting $\mathcal{I}$ denote the collection of all open subintervals of $[0, 1]$, we have the following

Theorem. The classes of strongly measurable $DP$-integrable and $DP^*$-integrable functions with values in $X$ coincide if and only if $X$ contains no isomorphic copy of $c_0$.

Proof. It is clear from the definition that every $DP^*$-integrable function is $DP$-integrable. So, suppose $X \nsubseteq c_0$ and that $f: [0, 1] \to X$ is strongly measurable and $DP$-integrable on $I \in \mathcal{I}$ with indefinite $DP$-integral $F$. Let $W \subseteq [0, 1]$ be perfect and $I' \subseteq I$, $I' \subseteq I$ with $I' \cap W \neq \emptyset$. Then since $f$ is $DP$-integrable on $I$, there is a subinterval $\bar{I} \subseteq I'$ with $\bar{I} \cap W \neq \emptyset$ such that given any interval $I'' \subseteq \bar{I}$, $x^*f$ is Lebesgue integrable on $I'' \cap W$ for all $x^* \in X^*$. Furthermore (by Corollary 1 to Theorem 3.5.3 of [2]), the strong measurability of $f$ yields that on any such $I'' \cap W$, $f$ can be represented in the form

$$f(t) = \sum_n x_n x_{E_n}(t) + g(t)$$

where $(x_n) \subseteq X$, $(E_n)$ is a measurable partition of $I'' \cap W$ and $g$ is bounded

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and strongly measurable. Now letting $E = I'' \cap W$, by the integrability of $x^* f$ on $E$, we have for all such $E$ and all $x^* \in X^*$,  

$$\sum_n |x^*(x_n) \lambda(E_n \cap E)| = \sum_n |x^*(\lambda(E_n \cap E)x_n)| < \infty.$$  

Hence the series $\sum_n \lambda(E_n \cap E)x_n$, for all $E = I'' \cap W$, $I'' \subseteq \bar{I}$, is weakly unconditionally Cauchy in $X$, and so, by the Bessaga-Pelczyński characterization of Banach spaces not containing $c_0$ [1], $\sum_n \lambda(E_n \cap E)x_n$ is unconditionally convergent in $X$ for all $I'' \cap W = E$, $I'' \subseteq \bar{I}$. Thus (see pp. 77–78 of [2]) $f$ is Pettis integrable on $\bar{I} \cap W$. Finally, by the uniqueness of the representing set function for $f$, $\int_{I''} F_W = (P) \int_{I''} f x_W d\lambda$ holds for all subintervals $I'' \subseteq \bar{I}$ and so $f$ is $DP^*$-integrable on $I$.

On the other hand, suppose $X \supseteq c_0$. Define the function $f: [0, 1] \to c_0$ by  

$$f(t) = (x_{[0,1]}(t), 2x_{[0,1/2]}(t), \ldots, nx_{[0,1/n]}(t), \ldots)$$  

for $t \in [0, 1]$. Then it can be readily established that $f$ is $DP$-integrable but not $DP^*$-integrable, and the theorem is complete.

**Corollary.** If $X$ is weakly sequentially complete then every strongly measurable, $X$-valued Denjoy-Gelfand integrable function is Denjoy-Pettis integrable.

**References**


**Department of Mathematics, Kent State University, Kent, Ohio 44242**

**Current address:** Department of Mathematics, Hiram College, Hiram, Ohio 44234