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SHORTER NOTES

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A NOTE ON THE DENJOY INTEGRABILITY OF ABSTRACTLY-VALUED FUNCTIONS

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ABSTRACT. We present necessary and sufficient conditions on a Banach space X that the classes of strongly measurable X-valued Denjoy-Pettis integrable and Denjoy-Gelfand integrable functions coincide.

In this note we give necessary and sufficient conditions on a Banach space X that the classes of strongly measurable, X-valued Denjoy-Pettis and Denjoy-Gelfand integrable functions coincide. We preserve here, for the most part, the notations of D. W. Solomon [3] and, for the sake of simplicity, restrict our attention to Lebesgue measure λ on $[0, 1]$.

Throughout, following Solomon, we let DP denote the Denjoy-Gelfand integral and DP^* the Denjoy-Pettis integral. All point functions f will be strongly measurable, defined on $[0, 1]$ and take values in the Banach space X. Letting $\mathcal{I}$ denote the collection of all open subintervals of $[0, 1]$, we have the following

THEOREM. *The classes of strongly measurable DP-integrable and DP^*-integrable functions with values in X coincide if and only if X contains no isomorphic copy of c_0.*

PROOF. It is clear from the definition that every DP^*-integrable function is DP-integrable. So, suppose $X \not\supseteq c_0$ and that $f\colon [0,1] \to X$ is strongly measurable and DP-integrable on $I \in \mathcal{I}$ with indefinite DP-integral F. Let $W \subseteq [0,1]$ be perfect and $I' \in \mathcal{I}$, $I' \subseteq I$ with $I' \cap W \neq \varnothing$. Then since f is DP-integrable on I, there is a subinterval $\tilde{I} \subseteq I'$ with $\tilde{I} \cap W \neq \varnothing$ such that given any interval $I'' \subseteq \tilde{I}$, x^*f is Lebesgue integrable on $I'' \cap W$ for all $x^* \in X^*$. Furthermore (by Corollary 1 to Theorem 3.5.3 of [2]), the strong measurability of f yields that on any such $I'' \cap W$, f can be represented in the form

$$f(t) = \sum_n x_n \chi_{E_n}(t) + g(t)$$

where $(x_n) \subseteq X$, (E_n) is a measurable partition of $I'' \cap W$ and g is bounded

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and strongly measurable. Now letting $E = I'' \cap W$, by the integrability of x^*f on E, we have for all such E and all $x^* \in X^*$,

$$\sum_n |x^*(x_n)\lambda(E_n \cap E)| = \sum_n |x^*(\lambda(E_n \cap E)x_n)| < \infty.$$

Hence the series $\sum_n \lambda(E_n \cap E)x_n$, for all $E = I'' \cap W$, $I'' \subseteq \tilde{I}$, is weakly unconditionally Cauchy in X, and so, by the Bessaga-Pelczynski characterization of Banach spaces not containing c_0 [1], $\sum_n \lambda(E_n \cap E)x_n$ is unconditionally convergent in X for all $I'' \cap W = E$, $I'' \subseteq \tilde{I}$. Thus (see pp. 77–78 of [2]) f is Pettis integrable on $\tilde{I} \cap W$. Finally, by the uniqueness of the representing set function for f, $\int_{I''} F_W = (P)\int_{I''} f\chi_W d\lambda$ holds for all subintervals $I'' \subseteq \tilde{I}$ and so f is DP^*-integrable on I.

On the other hand, suppose $X \supseteq c_0$. Define the function f: $[0, 1] \to c_0$ by

$$f(t) = (\chi_{[0,1]}(t), 2\chi_{[0,1/2]}(t), \ldots, n\chi_{[0,1/n]}(t), \ldots)$$

for $t \in [0, 1]$. Then it can be readily established that f is DP-integrable but not DP^*-integrable, and the theorem is complete.

COROLLARY. *If X is weakly sequentially complete then every strongly measurable, X-valued Denjoy-Gelfand integrable function is Denjoy-Pettis integrable.*

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