PLURIHARMONIC FUNCTIONS IN BALLS

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Abstract. It is proved that a function is pluriharmonic in the open unit ball of \( \mathbb{C}^n \) if and only if it is harmonic with respect to both the ordinary Laplacian and the invariant Laplace-Beltrami operator.

A complex function \( u \) defined in an open subset of \( \mathbb{C}^n \) is said to be \textit{pluriharmonic} if it satisfies the \( n^2 \) partial differential equations

\[
\frac{\partial^2 u}{\partial z_i \partial \overline{z}_j} = 0 \quad (i, j = 1, \ldots, n).
\]

The importance of this system of equations stems from the fact that a real function is pluriharmonic if and only if it is locally the real part of a holomorphic function [1, p. 271].

The Laplacian \( \Delta u \) of \( u \) is

\[
\Delta u = 4 \sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \overline{z}_k}.
\]

In the open unit ball \( B \) of \( \mathbb{C}^n \) we also have the so-called \textit{Laplace-Beltrami operator} \( \tilde{\Delta} \), defined by

\[
\tilde{\Delta} u = (1 - |z|^2) \left[ \Delta u - 4 \sum_{i,j=1}^{n} z_i \overline{z}_j \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j} \right],
\]

where \( |z|^2 = |z_1|^2 + \cdots + |z_n|^2 \). This is frequently called the \textit{invariant} Laplacian in \( B \), since

\[
\tilde{\Delta}(u \circ \phi) = (\tilde{\Delta} u) \circ \phi
\]

for every holomorphic one-to-one mapping \( \phi \) of \( B \) onto \( B \) [2, pp. 25–27].

It is perfectly obvious that every pluriharmonic \( u \) in \( B \) satisfies \( \Delta u = \tilde{\Delta} u = 0 \). The point of this note is to prove the converse.

Theorem. If \( u \) is a function in \( B \) that satisfies

\[
\Delta u = 0 \quad \text{and} \quad \tilde{\Delta} u = 0
\]

then \( u \) is pluriharmonic.
Proof. The equation $\Delta u = 0$ says that $u$ is harmonic in $B$. Hence $u$ can be expanded in a series

$$u = \sum_{k=0}^{\infty} P_k$$

in which each $P_k$ is a homogeneous harmonic polynomial of total degree $k$ in the variables $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$. The series converges uniformly on compact subsets of $B$, and successive derivatives of $u$ can be obtained by termwise differentiation of the series.

Define $\Lambda$ by

$$\Lambda f = \sum_{i,j=1}^{n} z_i \bar{z}_j \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}.$$ 

By (5), $\Lambda u = 0$. Since $\Lambda$ carries the class of all homogeneous polynomials of degree $k$ into itself, it follows from (6) that $\Lambda P_k = 0$ for $k = 0, 1, 2, \ldots$.

Each $P_k$ has a decomposition

$$P_k = \sum_{p+q=k} f_{p,q}$$

where $f_{p,q}$ has total degree $p$ in the variables $z_1, \ldots, z_n$, and has total degree $q$ in $z_1, \ldots, \bar{z}_n$. If $M$ is any monomial that occurs in $f_{p,q}$ then

$$M(z) = c z_1^{\alpha_1} \cdots z_n^{\alpha_n} \bar{z}_1^{\beta_1} \cdots \bar{z}_n^{\beta_n}$$

with $\alpha_1 + \cdots + \alpha_n = p, \beta_1 + \cdots + \beta_n = q$. A simple computation shows that $\Lambda M$ is $M$ times $\sum \alpha_i \beta_j = pq$. Thus

$$\Lambda f_{p,q} = pq f_{p,q}.$$ 

Since $\Lambda P_k = 0$, (10) implies that $pq f_{p,q} = 0$ for all $p$ and $q$. Hence $f_{p,q} = 0$ unless $p = 0$ or $q = 0$. It follows that $P_k = f_{k,0} + f_{0,k}$, the sum of a holomorphic polynomial and one whose complex conjugate is holomorphic. Thus each $P_k$ in (6) is pluriharmonic, and term-by-term differentiation shows the same for $u$.

Postscript. In view of this theorem, one may ask whether the inequalities

$$\Delta u > 0, \quad \tilde{\Delta} u > 0$$

imply that $u$ is plurisubharmonic in $B$. The polynomial

$$u(z) = z_1 \bar{z}_1 + z_2 \bar{z}_2 - z_3 \bar{z}_3$$
shows that this is not so (at least when \( n > 2 \)), since \( \Delta u = 4 \) and \( \bar{\Delta} u = 4(1 - |z|^2)(1 - u) > 0 \), but \( u(0,0,w) = -|w|^2 \) is not a subharmonic function in the unit disc.

REFERENCES


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