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PLURIHARMONIC FUNCTIONS IN BALLS

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ABSTRACT. It is proved that a function is pluriharmonic in the open unit ball of $\mathbf{C}^n$ if and only if it is harmonic with respect to both the ordinary Laplacian and the invariant Laplace-Beltrami operator.

A complex function u defined in an open subset of $\mathbf{C}^n$ is said to be *pluriharmonic* if it satisfies the n^2 partial differential equations

$$\partial^2 u/\partial z_i \partial \bar{z}_j = 0 \qquad (i,j = 1, \ldots, n). \tag{1}$$

The importance of this system of equations stems from the fact that a real function is pluriharmonic if and only if it is locally the real part of a holomorphic function [1, p. 271].

The *Laplacian* Δu of u is

$$\Delta u = 4 \sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k}. \tag{2}$$

In the open unit ball B of $\mathbf{C}^n$ we also have the so-called *Laplace-Beltrami operator* $\tilde{\Delta}$, defined by

$$\tilde{\Delta} u = (1 - |z|^2)\left[\Delta u - 4 \sum_{i,j=1}^{n} z_i \bar{z}_j \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right], \tag{3}$$

where $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$. This is frequently called the *invariant* Laplacian in B, since

$$\tilde{\Delta}(u \circ \phi) = (\tilde{\Delta} u) \circ \phi \tag{4}$$

for every holomorphic one-to-one mapping ϕ of B onto B [2, pp. 25–27].

It is perfectly obvious that every pluriharmonic u in B satisfies $\Delta u = \tilde{\Delta} u = 0$. The point of this note is to prove the converse.

THEOREM. *If u is a function in B that satisfies*

$$\Delta u = 0 \quad \textit{and} \quad \tilde{\Delta} u = 0 \tag{5}$$

then u is pluriharmonic.

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PROOF. The equation $\Delta u = 0$ says that u is harmonic in B. Hence u can be expanded in a series

$$u = \sum_{k=0}^{\infty} P_k \tag{6}$$

in which each P_k is a homogeneous harmonic polynomial of total degree k in the variables $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$. The series converges uniformly on compact subsets of B, and successive derivatives of u can be obtained by termwise differentiation of the series.

Define Λ by

$$\Lambda f = \sum_{i,j=1}^{n} z_i \bar{z}_j \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}. \tag{7}$$

By (5), $\Lambda u = 0$. Since Λ carries the class of all homogeneous polynomials of degree k into itself, it follows from (6) that $\Lambda P_k = 0$ for $k = 0, 1, 2, \ldots$.

Each P_k has a decomposition

$$P_k = \sum_{p+q=k} f_{p,q} \tag{8}$$

where $f_{p,q}$ has total degree p in the variables $z_1, \ldots, z_n$, and has total degree q in $\bar{z}_1, \ldots, \bar{z}_n$. If M is any monomial that occurs in $f_{p,q}$ then

$$M(z) = c z_1^{\alpha_1} \cdots z_n^{\alpha_n} \bar{z}_1^{\beta_1} \cdots \bar{z}_n^{\beta_n} \tag{9}$$

with $\alpha_1 + \cdots + \alpha_n = p$, $\beta_1 + \cdots + \beta_n = q$. A simple computation shows that ΛM is M times $\sum \alpha_i \beta_j = pq$. Thus

$$\Lambda f_{p,q} = pq f_{p,q}. \tag{10}$$

Since $\Lambda P_k = 0$, (10) implies that $pq f_{p,q} = 0$ for all p and q. Hence $f_{p,q} = 0$ unless $p = 0$ or $q = 0$. It follows that $P_k = f_{k,0} + f_{0,k}$, the sum of a holomorphic polynomial and one whose complex conjugate is holomorphic. Thus each P_k in (6) is pluriharmonic, and term-by-term differentiation shows the same for u.

Postscript. In view of this theorem, one may ask whether the inequalities

$$\Delta u > 0, \qquad \tilde{\Delta} u > 0$$

imply that u is plurisubharmonic in B. The polynomial

$$u(z) = z_1 \bar{z}_1 + z_2 \bar{z}_2 - z_3 \bar{z}_3$$

shows that this is not so (at least when $n > 2$), since $\Delta u = 4$ and $\tilde{\Delta} u = 4(1 - |z|^2)(1 - u) > 0$, but $u(0, 0, w) = -|w|^2$ is not a subharmonic function in the unit disc.

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