PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 62, Number 1, January 1977

ON DERIVATION ALGEBRAS OF MALCEV ALGEBRAS

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ABSTRACT. It is shown that if A is a Malcev algebra over a field of characteristic 0, then A is semisimple if and only if the derivation algebra $\mathfrak{D}(A)$ is semisimple. It is then shown that A is semisimple if and only if $A^* = \mathfrak{L}(A) + \mathfrak{D}(A)$ is semisimple, where $\mathfrak{L}(A)$ is the Lie multiplication algebra of A.

Let A be a nonassociative algebra over a field of characteristic 0 and let $\mathfrak{D}(A)$ be the derivation algebra of A. For certain classes of algebras (Lie, Jordan, alternative), there are results linking the semisimplicity of A to that of $\mathfrak{D}(A)$. Although some results of a related nature are available for Malcev algebras, it has not yet been shown that the semisimplicity of A and $\mathfrak{D}(A)$ are equivalent. It is the purpose of the present note to obtain this result and several related ones.

All algebras discussed here will be finite dimensional over a field of characteristic 0.

THEOREM 1. *Let A be a Malcev algebra over a field F of characteristic* 0. *Then A is semisimple if and only if $\mathfrak{D}(A)$ is semisimple.*

PROOF. Suppose that $\mathfrak{D}(A)$ is semisimple. By [11, Theorem 1] the radical $R(A)$ of A is contained in the center $Z(A)$ of A. Then by [10, Theorem 1] A is the direct sum of $R(A)$ and a maximal semisimple subalgebra $S(A)$ of A. Then $\mathfrak{D}(A)$ is the direct sum of ideals $\mathfrak{D}_1$ and $\mathfrak{D}_2$ where

$$\mathfrak{D}_1 = \{D \in \mathfrak{D}(A); D\colon S(A) \to S(A), D\colon R(A) \to 0\}$$

and

$$\mathfrak{D}_2 = \{D \in \mathfrak{D}(A); D\colon S(A) \to 0\}$$

(see the proof of [11, Theorem 1]). Suppose that $R(A) \neq 0$. Let T be the projection of A onto $R(A)$ with null space $S(A)$. Since $0 \neq R(A) \subseteq Z(A)$, $T \in \mathfrak{D}_2$ and $T \neq 0$. Let $T_1 \in \mathfrak{D}_2$, $x \in R(A)$ and $y \in S(A)$. Then, since $R(A)$ is $\mathfrak{D}(A)$-invariant by [3, Theorem 14],

Received by the editors December 17, 1975 and, in revised form, June 8, 1976.
AMS (MOS) subject classifications (1970). Primary 17A30.

$$(x + y)[T, T_1] = (x + y)TT_1 - (x + y)T_1T = xT_1 - xT_1 = 0.$$

Hence $T \in Z(\mathfrak{D}_2)$ and since $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are ideals in $\mathfrak{D}(A)$, $T \in Z(\mathfrak{D}(A))$. This contradicts the semisimplicity of $\mathfrak{D}(A)$, hence $R(A) = 0$.

Conversely let K be an algebraic closure of F. Then $\mathfrak{D}(A_K) \simeq \mathfrak{D}(A)_K$ (see [6, p. 233]). Semisimplicity of a Malcev algebra is equivalent to nondegeneracy of the Killing form [5, Theorem A] and the latter is preserved under extension of the base field, hence A_K is semisimple. By [5, Korollar 1], A_K is the direct sum of simple ideals. Each of these simple ideals is either a simple Lie algebra or is a 7 dimensional algebra obtained from a Cayley algebra as in [8, pp. 433–435]. This follows from combined results of Sagle [9] and Loos [5, Theorem B]. In either case, the simple algebra has simple derivation algebra. In particular, the non-Lie simple algebra has the exceptional simple Lie algebra G_2 for its derivation algebra [8, p. 455]. It follows that $\mathfrak{D}(A_K)$ is semisimple, hence $\mathfrak{D}(A)$ is also.

In order to extend a classical result of Leger and Togo [4] to general algebras, Ravisankar [6] considered $A^* = \mathfrak{L}(A) + \mathfrak{D}(A)$ where $\mathfrak{L}(A)$ is the Lie multiplication algebra of A. $\mathfrak{L}(A)$ is an ideal of A^* [6, p. 225]. We now show that the result of Theorem 1 holds with $\mathfrak{D}(A)$ replaced by A^*. In order to do this we need the following result.

Lemma. *Let A be a Malcev algebra over a field of characteristic* 0. *Then $\mathfrak{L}(A)$ is semisimple or* 0 *if and only if $R(A) \subseteq Z(A)$.*

Proof. If $R(A) \subseteq Z(A)$, then $R(A)$ is complemented by a semisimple subalgebra $S(A)$ by [10, Theorem 1]. If $S(A) = 0$, then $\mathfrak{L}(A) = 0$. Suppose that $S(A) \neq 0$. Then $S(A)$ is $\mathfrak{L}(A)$ invariant since $R(A) \subseteq Z(A)$, and the restriction mapping is an isomorphism from $\mathfrak{L}(A)$ to $\mathfrak{L}(S(A))$. Since $S(A)$ is semisimple, $\mathfrak{L}(S(A))$ is also [8, Corollary 7.3].

Conversely if $\mathfrak{L}(A)$ is semisimple, then, using a theorem of Weyl [1, Théorème 2, p. 75], $Z(A)$ is complemented by an ideal B of A. Clearly $Z(B) = 0$ and $\mathfrak{L}(A) \simeq \mathfrak{L}(B)$. Hence B is semisimple by [8, Theorem 7.2] and the result holds. If $\mathfrak{L}(A) = 0$, then $Z(A) = A$.

Theorem 2. *Let A be a Malcev algebra over a field of characteristic* 0. *Let $A^* = \mathfrak{L}(A) + \mathfrak{D}(A)$. Then A^* is semisimple if and only if A is semisimple.*

Proof. If A is semisimple, then $\mathfrak{D}(A)$ and $\mathfrak{L}(A)$ are semisimple. Then $A^*/\mathfrak{L}(A) \simeq \mathfrak{D}(A)/\mathfrak{D}(A) \cap \mathfrak{L}(A)$ is semisimple by [1, Corollaire 2, p. 76]. Since $R(A^*)$ projects onto the radical of $A^*/\mathfrak{L}(A)$ [1, Corollaire 3, p. 76], $R(A^*) \subseteq \mathfrak{L}(A)$. Hence $R(A^*) = 0$ and A^* is semisimple.

Conversely, if A^* is semisimple, then $\mathfrak{L}(A)$ is semisimple or $\mathfrak{L}(A) = 0$, hence $R(A) \subseteq Z(A)$ and $R(A)$ is complemented by a semisimple subalgebra $S(A)$. Suppose $R(A) \neq 0$. By [11, Theorem 1], $\mathfrak{D}(A)$ acts completely reducibly on A, hence $\mathfrak{D}(A) = S \oplus Z$ where S is a semisimple subalgebra of $\mathfrak{D}(A)$ and Z is the center of $\mathfrak{D}(A)$. Since $A^*/\mathfrak{L}(A)$ is semisimple, $Z \subseteq \mathfrak{L}(A)$. Let T be the projection of A on $R(A)$ with null space $S(A)$. Then $T \in \mathfrak{D}(A)$. Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$

be as in the proof of Theorem 1. Then both $R(A)$ and $S(A)$ are invariant under $\mathfrak{D}_1 \oplus \mathfrak{D}_2 = \mathfrak{D}(A)$, hence T commutes with each element of $\mathfrak{D}(A)$. Therefore $T \in Z \subseteq \mathfrak{L}(A)$. Since $R(A) \subseteq Z(A)$, $\mathfrak{L}(A)$ annihilates $R(A)$. Hence $T = 0$ and $R(A) = 0$.

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