A CURIOSITY CONCERNING THE DEGREES OF THE CHARACTERS OF A FINITE GROUP

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Abstract. Let $G$ be a finite group with irreducible characters $\{\ldots, \chi, \ldots\}$ and $K = \mathbb{Q}(\ldots, \chi, \ldots)$ the field generated over the rationals by their values. We will prove:

If $K = \mathbb{Q}$ (or if $[K: \mathbb{Q}]$ is odd) then \[\prod_{\chi(1) \text{ odd}} \chi(1) \text{ is a perfect square.}\]

More generally,

Theorem. (a) \[(-1)^{(\Sigma \chi(1) -(m+1))/2} \prod_{\chi(1) \text{ odd}} \chi(1)\]

is a square in $K$ if $|G|$ is even.

(b) $(-1)^{(|G|-1)/2}$ is a square in $K$ if $|G|$ is odd.

[Recall $|G| = \Sigma \chi(1)^2$ so that $|G| \equiv \Sigma \chi(1) \mod 2$, and so $\Sigma \chi(1) \equiv m + 1 \mod 2$ where $m$ denotes the number of involutions in $G$.]

Proof of Theorem. Let $F$ be any splitting field for $G$ such that $\text{char } F \nmid |G|$. Consider the $F$-vector space $FG$ on which we have the nondegenerate symmetric bilinear form defined by $B(g, h) = \rho(gh)$ for $g, h \in G$, where $\rho$ is the trace of the regular representation of $G$. So

\[B(g, h) = \begin{cases} 0, & g \neq h^{-1}, \\ |G|, & g = h^{-1}, \end{cases}\]

and with respect to this basis of group elements, $B$ is the direct sum of $(|G| - (m + 1))/2$-matrices $(\begin{smallmatrix} 0 & |G| \\ |G| & 0 \end{smallmatrix})$ and $m + 1 (1 \times 1)$-matrices $(|G|)$. So the discriminant of $B$ is $(-1)^{(|G|-1)/2}|G|^{|G|}$. But, if we identify $FG$ with the direct sum $\sum M_{\chi(1)}(F)$ of $(\chi(1) \times \chi(1))$ matrix algebras over $F$, and if $e_{ij}^\chi$ denote the matrix units, then

\[B(e_{ij}^\chi, e_{kl}^\lambda) = \begin{cases} \chi(1), & i = l, j = k, \chi = \lambda, \\ 0, & \text{otherwise}. \end{cases}\]

So with respect to this basis, $B$ is the direct sum of $(\chi(1)^2 - \chi(1))/2$-matrices $(\begin{smallmatrix} 0 & \chi(1) \\ \chi(1) & 0 \end{smallmatrix})$ and $\chi(1) (1 \times 1)$-matrices $(\chi(1))$, and so has discriminant

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\[ \prod_{\chi} (-1)^{(\chi(1)^2 - \chi(1))/2} (\chi(1)^2 - \chi(1))/2 \chi(1)^{\chi(1)} \]

\[ = (-1)^{|G| - \Sigma \chi(1)^2}/2 \prod \chi(1)^{\chi(1)^2}. \]

Hence

\[ (-1)^{|G| - (m + 1)/2} |G|^{|G|} \quad \text{and} \quad (-1)^{|G| - \Sigma \chi(1)^2}/2 \prod \chi(1)^{\chi(1)^2} \]

differ by a square in \( F \).

If \( |G| \) is odd, then the nontrivial characters occur in conjugate pairs, so that \( \prod \chi(1)^{\chi(1)^2} \) is a square; moreover, since \( \Sigma \chi(1) = 1 + 2 \Sigma_{c=1}^{c-1/2} (2k + 1) \), where \( c \) denotes the number of conjugacy classes,

\[ \sum \chi(1) \equiv c \mod 4. \]

Also, (Burnside) \( |G| \equiv c \mod 16 \), so we have that \( (-1)^{|G| - 1}/2 |G| \) is a square in \( F \).

If \( \text{char } F = 0 \), then the intersection of all splitting fields \( F \) is just \( K \). [Given a simple algebra finite dimensional over a number field \( L \), the Grunwald-Wang and Tchebotarev Density Theorems imply the existence of a prime \( p \) in \( L \) and maximal subfields \( F_1 \) and \( F_2 \) such that \( p \) splits completely in \( F_1 \) but is divisible by only one prime of \( F_2 \); hence \( F_1 \cap F_2 = L \). An analogous result holds for a semisimple algebra all of whose simple components have the same center; in the case of a group algebra, this common center can be chosen to be \( K \) above], and so our Theorem follows.

**Remarks.**

1. If all the characters of \( G \) are real, we must have

\[ (-1)^{(\Sigma \chi(1) - (m + 1))/2} = 1, \quad \text{i.e.,} \quad \sum \chi(1) \equiv m + 1 \mod 4. \]

2. If \( F \) can be chosen to be real, we have that

\[ \sum \frac{\chi(1)^2 - \chi(1)}{2} = \frac{|G| - (m + 1)}{2} = \text{Witt index of } B, \]

and so \( \Sigma \chi(1) = m + 1 \).

Both of these facts follow also from the classical Frobenius-Schur count of involutions.

3. If \( G \) is cyclic of odd order \( n \), then Theorem (b) states that

\[ (-1)^{(n - 1)/4} \sqrt{n} \in \mathbb{Q}(\xi_n). \]

4. Recall that for even \( |G| \), there are an odd number of nonprincipal, nonconjugate characters having both odd degree and an odd number of conjugates [Proc. Amer. Math. Soc. 30 (1971), 247–248]. In other words, \( \prod_{\chi(1) \text{ odd}} \chi(1) \) is nontrivial, and is never square solely by virtue of conjugate characters.

5. It would be of interest to find necessary and sufficient conditions on \( G \) to insure that every character has an odd number of conjugates (this implies
that the characters are real) i.e., that $[K : \mathbb{Q}]$ is odd. It would also be of interest to determine exactly the discriminant of $K$. Also, it is unknown which abelian extensions of $\mathbb{Q}$ appear as $K = \mathbb{Q}(\ldots, \chi, \ldots)$ for some finite group $G$.

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