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ON A PROBLEM OF BRUCKNER AND CEDER CONCERNING THE SUM OF DARBOUX FUNCTIONS

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ABSTRACT. The main purpose of this paper is to show that for some continuous function f and any preassigned, countable and dense set D of real numbers there exists a measurable function d which takes on every real value in every interval such that the range of $f + d$ is D.

A real valued function f defined on an interval I is said to have the intermediate value property if whenever x_1 and x_2 are in I, and y is any number between $f(x_1)$ and $f(x_2)$, there is a number x_3 between x_1 and x_2 such that $f(x_3) = y$. Because of Darboux's work on this property, one now usually calls a function having the intermediate value property a Darboux function. A function f is Darboux if and only if f maps any connected subset of I onto a connected set.

Since the sum of Darboux functions may fail to be a Darboux function, some mathematicians have examined how badly it can fail. Bruckner and Ceder [1] have recently shown that corresponding to each Darboux function f which is not constant on any subinterval, there exists a function d which takes on every real value in every subinterval of I such that the range of $f + d$ is a preassigned countable and dense set. Certainly d is a Darboux function and the range of $f + d$ is totally disconnected.

The technique used by Bruckner and Ceder in constructing the function d does not imply the measurability of d. Indeed Bruckner and Ceder have shown that d must be nonmeasurable whenever f is absolutely continuous. They asked whether d had to be nonmeasurable if we weaken absolute continuity of f to continuity of f.

This paper answers this question affirmatively for certain continuous functions f. This is a consequence of the following.

THEOREM. *Let g be a Darboux function which is not constant on any subinterval of its domain I such that the set*

$$A = \{\alpha\colon \alpha \in R, g^{-1}(\alpha) \textit{ is perfect}\}$$

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is dense in $g(I)$. Then there exists a homeomorphism h from I onto itself such that for every countable, dense set $D \subset R$ there exists a function d which takes on every real value in every subinterval of I such that D is the range of $(g \circ h) + d$ and $(g \circ h) + d$ is constant almost everywhere.

It is known that there exist continuous functions g which are not constant on any interval such that $A = g(I)$. See for example [4]. (The properties of the set A for a Darboux function g are discussed in [2] and [3].) Let g be such a function. Now define f by $f = g \circ h$ where h is the homeomorphism of the Theorem above. Then f is continuous and from this Theorem the range of $f + d$ is D. By the Theorem $f + d$ is constant almost everywhere. Thus d is measurable.

In order to prove the Theorem we shall use the type of construction suggested by Theorem 1 of [1]. But first we need the following.

LEMMA. *Suppose $F \subset I$, and for every interval $J \subset I$ there exists a perfect set $P \subset F \cap J$. Then there exists a homeomorphism χ from I onto itself such that $\chi(F)$ is measurable and $|\chi(F)| = |I|$.*

PROOF. The Lemma is trivial if $F = I$. Suppose that $I \backslash F \neq \emptyset$. Let P^0 be a perfect subset of F. Choose an increasing homeomorphism h_1 from I onto itself so that $|h_1(P^0)| > \frac{1}{2}|I|$. Let $\{(a_i, b_i)\}$ be a sequence which may be finite, of disjoint open intervals, such that $I \backslash (\{a\} \cup \{b\} \cup h_1(P^0)) = \bigcup_i (a_i, b_i)$. Choose a sequence $\{I_i\}$ of nonoverlapping intervals so that $\bigcup_i I_i = \bigcup_i \langle a_i, b_i \rangle$ and

$$(1) \qquad |I_i| < \tfrac{1}{2}|I|.$$

Now choose for each i a perfect set $P_i \subset h_1^{-1}(I_i) \cap F$. Next choose a strictly increasing continuous function h_2 from I onto itself such that

$$(2) \qquad x \in h_1(P_0) \text{ implies } h_2(x) = x,$$

$$(3) \qquad h_2(I_i) = I_i$$

and $|h_2(h_1(P_i))| > \frac{1}{2}|I_i|$.

By (2) $x \in P^0$ implies $h_1(x) = h_2(h_1(x))$. Now put $P^1 = P^0 \cup \bigcup_i P_i$. Then P^1 is a perfect set, $P^0 \subset P^1 \subset F$, $|h_2(h_1(P^1))| > \frac{3}{4}|I|$ and by (2), (3) and (1), $|x - h_2(x)| < \frac{1}{2}|I|$.

Using this method we can define by induction the sequences $\{P^n\}$ of perfect sets and $\{h_n\}$ of strictly increasing continuous functions from I onto itself such that

$$x = h_2(x) = h_3(x) = \cdots \qquad \text{for } x \in h_1(P^0),$$
$$x = h_3(x) = h_4(x) = \cdots \qquad \text{for } x \in h_2(h_1(P^1))$$

and generally,

$$(4) \quad x = h_{n+1}(x) = h_{n+2}(x) = \cdots \quad \text{for } x \in h_n(h_{n-1}(\cdots h_1(P^{n-1})\cdots)),$$

(5) $$|x - h_n(x)| < (1/2^{n-1})|I|,$$

(6) $$|h_n(h_{n-1}(\cdots h_1(P^{n-1})\cdots))| > (2^n - 1/2^n)|I|.$$

By (5) we have $|h_n(x) - h_{n+1}(h_n(x))| < (1/2^n)|I|$. Hence

$$|h_n(h_{n-1}(\cdots h_1(x)\cdots)) - h_{n+1}(h_n(\cdots h_1(x)\cdots))| < (1/2^n)|I|.$$

By this inequality the sequence of superpositions $\{h_n(h_{n-1}(\cdots h_1(x)\cdots))\}$ converges uniformly to a continuous function. Denote this limit function by χ. All h_n are strictly increasing so χ is increasing. By (4),

(7) $$\chi(x) = h_n(h_{n-1}(\cdots h_1(x)\cdots)) \quad \text{for } x \in P^{n-1}.$$

Hence χ is strictly increasing on P^{n-1} and also on the set $\bigcup_{n=1}^{\infty} P^n$ which is dense in I. Thus χ is strictly increasing. By (7)

$$\chi(P^{n-1}) = h_n(h_{n-1}(\cdots h_1(P^{n-1})\cdots)).$$

Hence, since $\chi(P^{n-1}) \subset \chi(F) \subset I$ by (6) we have $|\chi(E)| > ((2^n - 1)/2^n)|I|$. Hence $|\chi(F)| = |I|$.

This completes the proof of the Lemma.

PROOF OF THE THEOREM. Let $\{I_n\}$ be a sequence of all the open intervals with rational endpoints contained in I. Since $g(I_1)$ is a subinterval of $g(I)$ and A is dense in $g(I)$, there exists a point $\alpha_1 \in A \cap g(I_1)$. Let F_1 be a perfect set such that $F_1 \subset I_1 \cap g^{-1}(\alpha_1)$ and F_1 is nowhere dense in $g^{-1}(\alpha_1)$. Suppose we have chosen for each $i = 1, 2, \ldots, k$ perfect sets F_i and numbers α_i such that $F_i \subset I_i$, $g(F_i) = \alpha_i \in A$, F_i is nowhere dense in $g^{-1}(\alpha_i)$ and the sets F_i are disjoint. Let I^* be an interval such that $I^* \subset I_{k+1}$ and $I^* \cap (\bigcup_{i=1}^{k} F_i) = \varnothing$. Choose a perfect set $F_{k+1} \subset I^*$ such that $g(F_{k+1}) = \alpha_{k+1} \in A$, F_{k+1} is nowhere dense in $g^{-1}(\alpha_{k+1})$. Then $F_{k+1} \cap F_i = \varnothing$ for $i \neq k + 1$. Clearly the sum $F = \bigcup_{k=1}^{\infty} F_k$ satisfies the conditions of the Lemma. Then there is a homeomorphism χ from I onto itself such that $|\chi(F)| = |I|$. Denote $H = I \setminus \chi(F)$, $h = \chi^{-1}$ and $\Gamma = \{(x,y)\colon x \in H, y + g(h(x)) \in D\}$. Certainly $\chi(F)$ is a set of the first category of Baire. Thus H is dense in I.

If B is a subset of $H \times R$ we shall denote its x-projection, i.e. the set $\{x\colon \exists_y (x,y) \in B\}$ by $\operatorname{dom} B$, and its y-projection, i.e. the set $\{y\colon \exists_x (x,y) \in B\}$ by $\operatorname{rng} B$. For any $t \in H$ we shall denote $B_t = \{(x,y)\colon x = t, (x,y) \in B\}$ and for any $s \in R$ we shall denote $B^s = \{(x,y)\colon y = s, (x,y) \in B\}$. Since Γ is the union of the graphs of all functions $z_n - g(h(x))$ where $z_n \in D$, it follows that $\operatorname{rng} \Gamma = R$.

We shall say that two points $A(x',y')$ and $B(x'',y'')$ of Γ are equivalent, $A \sim B$, if there exists a finite sequence of points $A_i(x_i,y_i) \in \Gamma$, $i = 1, 2, \ldots, n$, such that $A = A_1$, $B = A_n$, and for each i we have either $x_i = x_{i+1}$ or $y_i = y_{i+1}$. Denote by $\Gamma/\!\sim$ the family of all equivalence classes for this equivalence relation. It is clear that for each $t \in H$, $\operatorname{rng} \Gamma_t$ contains a translation of D. Then $\operatorname{rng} \Gamma_t$ is a dense set. It follows from the definition of equivalence that if G is an equivalence class, then $G \cap \Gamma_t \neq \emptyset$ implies $G \supset \Gamma_t$. In the same way if $G \cap \Gamma^s \neq \emptyset$ then $G \supset \Gamma^s$.

Now we shall prove that all $\operatorname{dom}\Gamma^s$ are dense in H. Since g is not constant on any interval then for any open interval $I^* \subset I$ there exist points x' and x'' in I^* for which $g(h(x')) \neq g(h(x''))$. Since D is dense there exists an $r \in D$ such that $r - s$ lies between $g(h(x'))$ and $g(h(x''))$. Since $g \circ h$ is Darboux there exists an $x \in I^*$ for which $g(h(x)) = r - s$. If $x \in H$ then $(x,s) \in \Gamma$ and therefore $x \in \operatorname{dom}\Gamma^s \cap I^*$.

If $x \notin H$ then $x \in \chi(F)$. Thus $r - s = g(h(x)) = g(\chi^{-1}(x)) = \alpha_i \in A$. The level set $g^{-1}(\alpha_i)$ is a perfect set and $h(F) \cap g^{-1}(\alpha_i) = h(F_i) \cap g^{-1}(\alpha_i)$. The set $h(F_i)$ is nowhere dense in $g^{-1}(\alpha_i)$. It follows that there exists a point $x^* \in I^* \cap [g^{-1}(\alpha_i)\backslash h(F)]$. Thus $x^* \in H \cap I^*$ and $\alpha_i = g(h(x^*)) = r - s$. Hence $(x^*,s) \in \Gamma$ and $x^* \in \operatorname{dom}\Gamma^s \cap I^*$. This shows that $\operatorname{dom}\Gamma^s$ is dense in H.

Suppose $A(x',y') \sim B(x'',y'')$. Then

$$y'' - y' = \sum_{i=1}^{n-1} (y_{i+1} - y_i) = \sum_{i'} (y_{i'+1} - y_{i'})$$

where $1 \leqslant i' \leqslant n$ and $y_{i'+1} - y_{i'} \neq 0$. This inequality implies $x_{i'} = x_{i'+1}$. Then

$$y_{i'+1} - y_{i'} = y_{i'+1} - g(h(x_{i'+1})) - (y_{i'} - g(h(x_{i'}))).$$

So $y_{i'+1} - y_{i'}$ is the difference of two members of D. The set of all such differences is countable. The set of finite sums of these differences is countable too. Each difference $y'' - y'$ is such a sum of this form. Therefore the set of all differences of ordinates of points belonging to an equivalence class G is countable. Thus $\operatorname{rng} G$ is countable. Let us enumerate members of $\operatorname{rng} G$ as $\{y_m\}$ and all rational subintervals of I as $\{J_n\}$. Now enumerate all pairs (y_m, J_n) as $\{(y_{m_i}, J_{n_i})\}$. Since $\operatorname{dom}\Gamma^{y_{m_1}}$ is dense in H, there exists a point $t_i \in J_{n_1} \cap \operatorname{dom}\Gamma^{y_{m_1}}$. Let $t_1, t_2, \ldots, t_k$ be chosen such that $t_i \in J_{n_i} \cap \operatorname{dom}\Gamma^{y_{m_i}}$ and $t_i \neq t_j$ for $1 \leqslant i < j \leqslant k$. Take $t_{k+1} \in J_{n_{k+1}} \cap \operatorname{dom}\Gamma^{y_{m_{k+1}}}$. Each set $\operatorname{dom}\Gamma^{y_m}$ contains a dense set of members of the sequence $\{t_n\}$. Set $d(x) = y_{m_i}$ for $x = t_i$. Then $\{d(x)\colon x = t_n \in J_m\} = \operatorname{dom} G$. Since $\bigcup_{G \in \Gamma/\sim} \operatorname{rng} G = R$ the function d defined on $\bigcup_{G \in \Gamma/\sim} \{t_n\}$ maps each intersection $H \cap J_m$ onto R. If $G' \neq G''$ then $\operatorname{dom} G' \cap \operatorname{dom} G'' = \emptyset$. Therefore there is no contradiction in our extension of d from the subset of $\operatorname{dom} G$ to the sum $\bigcup \operatorname{dom} G$.

At present d is defined on a subset of H. This subset is dense in I, therefore each next extension of d on I must be Darboux. Let us fix a point $c \in D$. Set $d(x) = c - g(h(x))$ for each point x at which we have not defined $d(x)$. Since $|H| = 0$, $g(h(x)) + d(x) = c$ almost everywhere.

We still have to prove that $d - (g \circ h)$ maps I onto D. Let G^0 be the equivalence class containing Γ^0. Then $D \in \operatorname{rng} G^0$. For each $x = t_n$ the point $(x, d(x))$ belongs to Γ. Thus $d(x) + g(h(x)) \in D$. For each $x \neq t_n$ we have

$d(x) = c - g(h(x))$ where $c \in D$ so that $d(x) + g(h(x)) \in D$. This completes the proof of Theorem.

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