PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 62, Number 1, January 1977

MEASURABLE, TAIL DISINTEGRATIONS OF THE HAAR INTEGRAL ARE PURELY FINITELY ADDITIVE[1]

LESTER E. DUBINS

ABSTRACT. There are countably additive probability measures, P, and sub-sigma fields, relative to which P admits no proper, measurable, conditional distributions, except, possibly, those which are purely finitely additive. The usual fair, coin-tossing probability measure and the tail sigma field illustrate this phenomenon. More generally, every measurable, disintegration of the Haar integral of any compact metrizable group, G, relative to the partition, Π, of G which consists of the left cosets of any dense denumerable subgroup S of G, or what comes to the same thing, relative to the sigma field of Haar-measurable subsets of G which are invariant under right translation by S, is purely finitely additive.

This note relates to [1] and [2], but is logically independent of these references.

Let S be a denumerable, dense subgroup of a compact, metrizable group, G, and let M be the unique, G-invariant mean defined on the space $C(G)$ of all real-valued continuous functions on G. Let $\Pi = G/S$ be the set of left cosets gS of S.

THEOREM 1. *Every measurable, Π-disintegration of M is purely finitely additive.*

To say that σ is a *measurable π-disintegration* of M means this. For all f in the domain of M,

$$Mf = \int \sigma_g(f^g)\, d(g) \tag{1}$$

where: (a) f^g is the trace of f on gS, that is, the restriction of f to gS; (b) σ_g is a mean which is supported by gS, so that $\sigma_g(gS)$ equals 1, $\sigma_g = \sigma_{g'}$ if $gS = g'S$, and σ_g is defined for f^g for all f in the domain of M, and (c) $\sigma_g(f^g)$ is integrable with respect to Haar measure dg.

If, except for a set of g's of Haar measure zero, σ_g is purely finitely additive, σ is said to be purely finitely additive.

The proof of Theorem 1 consists of two steps, the first of which is the

Received by the editors February 10, 1976.

AMS (MOS) subject classifications (1970). Primary 28A50, 60A05, 22C05.

Key words and phrases. Disintegrations, measures, conditional probability, compact groups, finite additivity, conglomerability.

[1] This research was prepared with the support of the National Science Foundation, Grant No. MPS 75-09459.

exhibition of a purely finitely additive disintegration $\hat{\sigma}$ of M, and the second is a demonstration that for every measurable disintegration σ of M, $\hat{\sigma}_g = \sigma_g$ for almost all g.

Since, for each g, gS is dense in G, the map $f \to f^g$ is a 1-1 mapping of $C(G)$ onto the space of uniformly continuous functions defined on gS. Consequently,

$$\hat{\sigma}_g(f_g) = Mf \tag{2}$$

defines $\hat{\sigma}$ uniquely. It is straightforward to verify that $\hat{\sigma}$ is a measurable Π-disintegration of M, henceforth called *elementary*.

Lemma 1. *The elementary* Π-*disintegration of* M *is purely finitely additive*.

Proof. Let $\varepsilon > 0$, and let $s_1, s_2, \ldots$ be an enumeration of the elements of S. As the Tietze extension theorem implies, there is, for each n, an $f_n \in C(G)$, $0 \leqslant f_n \leqslant 1$, such that $f_n(s_n)$ is 0 and f_n is less than 1 on an open set of Haar measure at most $\varepsilon 2^{-n}$. Let f_n' designate the infimum of f_i, $i = 1, \ldots, n$, and verify: $f_n' \in C(G)$; $f_n' \geqslant f_{n+1}'$; f_n' converges to 0 on S; and $M(f_n')$ exceeds $1 - \varepsilon$. Consequently, the restriction to S of the f_n' converges monotonely down to 0 everywhere on S and, for any $g \in S$, $\hat{\sigma}_g(f_n')$ exceeds $1 - \varepsilon$. This implies that, for $g \in S$, $\hat{\sigma}_g$ is purely finitely additive. By appropriately translating the sequence f_n, one concludes that every $\hat{\sigma}_g$ is purely finitely additive. Q.E.D.

A function ϕ defined on G is *S-invariant* if, for every $s \in S$, the right translate of ϕ by s is identical with ϕ. If, for every integrable, S-invariant ϕ, there is a constant c such that $\phi = c$ almost certainly, then *S acts ergodically*.

Lemma 2. *Every dense subset of G acts ergodically*.

Surely Lemma 2 is known, but since I know of no reference, I supply a proof.

Proof of Lemma 2. Let ϕ be integrable and S-invariant for the dense subset S of G. Let $f \in C(G)$ approximate ϕ in the L_1 norm. As shown by von Neumann [3], [4], there is a finite sequence of elements of G, $g_1, \ldots, g_n$, such that the average, F, of the right translates f_i of f by the g_i is uniformly close to a constant. Because $s_i \in S$ can be chosen arbitrarily close to g_i, it may be supposed that the g_i themselves are in S. Of course, the corresponding right translates ϕ_i of ϕ approximate f_i in the L_1 norm as well as ϕ approximates f. Consequently, the average, θ, of the ϕ_i approximates F, the average of the f_i. But, since ϕ is S-invariant, θ is ϕ. So F approximates ϕ in L_1, which implies that, for every $\varepsilon > 0$, there is a constant c such that $|\phi - c| \leqslant \varepsilon$ except on a set of measure less than ε. This implies that a single constant c satisfies this condition for all $\varepsilon > 0$. Now invoke countable additivity to see that $\phi = c$ almost surely. Q.E.D.

Lemma 3. *Let* σ *be a measurable*, Π-*disintegration of* M. *Then* $\sigma_g = \hat{\sigma}_g$ *for almost all* g.

PROOF. As implied by Lemma 2, for each $f \in C(G)$, there is a constant, $c(f)$, and a subset of G of measure 0, say $N(f)$, such that, for all g in the complement of $N(f)$,

$$\sigma_g(f^g) = c(f). \tag{3}$$

In view of (1), $c(f) = Mf$. Because G is compact and metrizable, there is a countable subset D of $C(G)$ which is dense in $C(G)$ for the topology of uniform convergence. Let N^c be the complement of the union of the null sets $N(f)$ for f in D. Summarizing, for all $g \in N^c$, for all $f \in D$,

$$\sigma_g(f^g) = Mf. \tag{4}$$

As is easily verified, for each g, the set of f for which (4) holds is closed in the uniform topology. Consequently, for $g \in N^c$, (4) holds for all $f \in C(G)$. Equivalently, for all $g \in N^c$, $\sigma_g = \hat{\sigma}_g$. Q.E.D.

In view of Lemmas 1 and 3, Theorem 1 is seen to hold.

Consider now the special case in which G is the product of a denumerable number of copies of the cyclic group of order two. The usual fair, coin-tossing, probability measure is identical with normalized Haar measure on G. And the atoms of the tail sigma field are simply the cosets of the subgroup S consisting of those elements of G which have no more than a finite number of nonzero coordinates. Hence, if Theorem 1 is applied to this example, one obtains

COROLLARY 1. *Let Ω be the space of infinite sequences of zeroes and ones, $\mathfrak{F}$ the field of finite-dimensional, Borel subsets of Ω, P the fair-coin probability measure on $\mathfrak{F}$. Then every Lebesgue measurable, proper, conditional distribution of P given the tail sigma field is purely finitely additive.*

A similar example is obtained by letting G be the unit interval, with addition taken mod 1, dg equal to Lebesgue measure, and $S \subset G$, the set of rationals.

Possibly the elementary Π-disintegration of M can be extended so as to be a Π-disintegration of all integrable functions, but the contrary seems to me to be more likely. Indeed, it is not unlikely that every Π-disintegrable measure defined for all Borel subsets of G is orthogonal to Haar measure on G. A related conjecture is that "measurable" can be deleted from the statement of Theorem 1.

REFERENCES

1. Lester E. Dubins and David Blackwell, *On existence and non-existence of proper, regular, conditional distributions*, Ann. Probability **3** (1975), 741–752.

2. Lester E. Dubins, *Finitely additive conditional probabilities, conglomerability and disintegrations*, Ann. Probability **3** (1975), 89–99. MR **50** # 11348.

3. John von Neumann, *Zum Haarschen Mass in topologischen Gruppen*, Compositio Math. **1** (1934), 106–114.

4. L. S. Pontryagin, *Topological groups*, 2nd ed., GITTL, Moscow, 1954; English transl., Gordon and Breach, New York, 1966. MR **17**, 171; **34** #1439.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720