PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 62, Number 1, January 1977

NEW SUBCLASSES OF THE CLASS OF CLOSE-TO-CONVEX FUNCTIONS

PRAN NATH CHICHRA

ABSTRACT. In this paper we introduce new subclasses of the class of close-to-convex functions. We call a regular function $f(z)$ an alpha-close-to-convex function if $(f(z)f'(z)/z) \neq 0$ for z in E and if for some nonnegative real number α there exists a starlike function $\phi(z) = z + \cdots$ such that

$$\operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}\right] > 0$$

for z in E.

We have proved that all alpha-close-to-convex functions are close-to-convex and have obtained a few coefficient inequalities for α-close-to-convex functions and an integral formula for constructing these functions.

Let $\mathfrak{F}_\alpha$ be the class of regular and normalised functions $f(z)$ which satisfy $\operatorname{Re}(f'(z) + \alpha z f''(z)) > 0$ for z in E. $f(z) \in \mathfrak{F}_\alpha$ gives $\operatorname{Re} f'(z) > 0$ for z in E provided $\operatorname{Re}\alpha \geqslant 0$. A sharp radius of univalence of the class of functions $f(z)$ for which $zf'(z) \in \mathfrak{F}_\alpha$ has also been obtained.

1. Recently Mocanu [4] introduced the concept of alpha-starlike functions. He called a function $f(z)$ an α-starlike function if $(f(z)f'(z)/z) \neq 0$ in $E = \{z/|z| < 1\}$ and if for some real number α,

$$\operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > 0 \tag{1.1}$$

for z in E. It is shown in [5] that all α-starlike functions are starlike in E. Moreover for $\alpha \geqslant 1$, all α-starlike functions are convex in E.

In this note we shall introduce new subclasses of the class of close-to-convex functions. We shall call $f(z) = z + a_2 z^2 + \cdots$ an alpha-close-to-convex function if $(f(z)f'(z)/z) \neq 0$ in E and if for some nonnegative real number α and for some starlike function $\phi(z) = z + \cdots$,

$$\operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}\right] > 0 \tag{1.2}$$

for z in E. For a fixed α we denote by C_α the class of α-close-to-convex functions.

In §2 we shall show that all α-close-to-convex functions are close-to-convex

Received by the editors May 26, 1975 and, in revised form, September 22, 1975.
AMS (MOS) subject classifications (1970). Primary 30A32.
Key words and phrases. Alpha-starlike functions, convex functions, starlike functions, close-to-convex functions, radius of univalence.

and $C_\alpha \subset C_\beta$ if $\alpha > \beta \geqslant 0$. It is interesting to note that C_∞ is the class of all convex functions in E. In §3 we find an integral representation formula and a few coefficient estimates for the class of α-close-to-convex functions. In §4 we shall define subclasses of the class of functions whose derivative has positive real part in E and study some of their properties.

2. To prove our main theorem we shall require the following lemma.

LEMMA 1. *Let $\alpha \geqslant 0$ and $D(z)$ be a starlike function in the open unit disc E. Let $N(z)$ be analytic in E and $N(0) = D(0) = 0 = N'(0) - 1 = D'(0) - 1$. Then* $\operatorname{Re}(N(z)/D(z)) > 0$ *for z in E whenever*

$$\operatorname{Re}\left[(1-\alpha)\frac{N(z)}{D(z)} + \alpha\frac{N'(z)}{D'(z)}\right] > 0 \tag{2.1}$$

for z in E.

The above lemma includes as a particular case a well-known and widely used lemma of Sakaguchi [10] which corresponds to the case $\alpha = 1$.

PROOF. Let an analytic function $\omega(z)$ be defined in E by

$$N(z)/D(z) = (1-\omega(z))/(1+\omega(z)). \tag{2.2}$$

Evidently $\omega(0) = 0$, $\omega(z) \neq -1$ for z in E. We shall now prove that $|\omega(z)| < 1$ for z in E. For if not, then by Jack's lemma [1] we could find ζ, $|\zeta| < 1$, such that $|\omega(\zeta)| = 1$ and

$$\zeta\omega'(\zeta) = k\omega(\zeta), \tag{2.3}$$

where $k \geqslant 1$.

We write

$$\psi(z) = (1-\alpha)\frac{N(z)}{D(z)} + \alpha\frac{N'(z)}{D'(z)}. \tag{2.4}$$

From (2.2), (2.3) and (2.4), we have

$$\psi(\zeta) = \frac{1-\omega(\zeta)}{1+\omega(\zeta)} - \frac{2\alpha k\omega(\zeta)}{(1+\omega(\zeta))^2}\frac{D(\zeta)}{\zeta D'(\zeta)}$$

with $|\omega(\zeta)| = 1$. Since

$$\operatorname{Re}\frac{1-\omega(\zeta)}{1+\omega(\zeta)} = 0, \qquad \operatorname{Re}\frac{D(\zeta)}{\zeta D'(\zeta)} > 0,$$

and $\omega(\zeta)/(1+\omega(\zeta))^2$ is real and positive, therefore $\operatorname{Re}\psi(\zeta) \leqslant 0$. This contradicts the given hypotheses that $\operatorname{Re}(\psi(z)) > 0$ for z in E. Then (2.2) gives $\operatorname{Re}(N(z)/D(z)) > 0$ for z in E.

On choosing $D(z) = \phi(z)$ and $N(z) = zf'(z)$ in the above lemma, we arrive at

THEOREM 1. *Let $f(z) \in C_\alpha$ and $\alpha \geqslant 0$. Then $f(z)$ is close-to-convex in E.*

THEOREM 2. *Let $\alpha > \beta \geqslant 0$. Then $C_\alpha \subset C_\beta$.*

PROOF. If $\beta = 0$, then $C_\alpha \subset C_0$, By Theorem 1. Let $\beta \neq 0$ and $f(z) \in C_\alpha$. Then there exists a starlike function $\phi(z)$ such that (1.2) holds. Therefore by Lemma 1,

$$\operatorname{Re}\,(zf'(z)/\phi(z)) > 0 \tag{2.5}$$

for z in E. From (1.2), (2.5) and the identity

$$(1-\beta)\frac{zf'(z)}{f(z)} + \beta\frac{(zf'(z))'}{\phi'(z)} = \frac{\beta}{\alpha}\left[\left(\frac{\alpha}{\beta}-1\right)\frac{zf'(z)}{\phi(z)} + (1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}\right],$$

we arrive at the desired result $f(z) \in C_\beta$.

3. THEOREM 3 (INTEGRAL REPRESENTATION FORMULA). *A function $f(z)$ is in C_α if and only if there exists a starlike function $\phi(z)$ and a function $P(z)$ which is regular and has positive real part in E such that*

$$f'(z) = \frac{1+c}{z[\phi(z)]^c}\int_0^z [\phi(\zeta)]^c\phi'(\zeta)P(\zeta)\,d\zeta \tag{3.1}$$

where $c = (1/\alpha) - 1$, $\alpha \neq 0$. If $\alpha = 0$, then

$$f'(z) = (\phi(z)/z)P(z). \tag{3.1$'$}$$

(*Powers in* (3.1) *are meant as principal values.*)

PROOF. Let $f(z) \in C_\alpha$ $(\alpha > 0)$. Then the quantity in brackets in (1.2) equals a function $P(z)$ with positive real part in E. Multiplying both sides of this equality by $\alpha^{-1}[\phi(z)]^c\phi'(z)$ we obtain

$$czf'(z)(\phi(z))^{c-1}\phi'(z) + (\phi(z))^c(zf'(z))' = (1+c)(\phi(z))^c\phi'(z)P(z). \tag{3.2}$$

The left-hand side of (3.2) is the exact differential of $zf'(z)(\phi(z))^c$. Therefore on integrating both sides of (3.2) with respect to z we obtain (3.1).

Conversely, if $f(z)$ satisfies (3.1), then it is easy to see that $f(z) \in C_\alpha$.

On choosing $\phi(z) = z/(1-z)^2$ and $P(z) = (1+z)/(1-z)$ in (3.1), we obtain the following function of C_α:

$$f(z) = \int_0^z (1+c)t^{-c-1}(1-t)^{2c}\left(\int_0^t \frac{\zeta^c(1+\zeta)^2}{(1-\zeta)^{2c+4}}d\zeta\right)dt \qquad (c = (1/\alpha) - 1). \tag{3.3}$$

The function (3.3) will serve as an extremal function for the next theorem.

THEOREM 4. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be in C_α. Then we have the sharp inequalities*:

(3.4)(i) $|a_2| \leqslant (2+\alpha)/(1+\alpha)$,

(3.5)(ii) $|a_3| \leqslant (9+23\alpha+6\alpha^2)/(3(1+\alpha)(1+2\alpha))$,

(3.6)(iii) $|a_4| \leqslant (4+22\alpha+34\alpha^2+6\alpha^3)/((1+\alpha)(1+2\alpha)(1+3\alpha))$.

PROOF. Let $f(z) = z + a_2 z^2 + \cdots \in C_\alpha$. Then there exists a starlike function $\phi(z) = z + b_2 z^2 + \cdots$ and a function $P(z) = 1 + c_1, z + c_2 z^2 + \cdots$ with positive real part in E such that the quantity in brackets in (1.2) equals $P(z)$. Multiplying both sides of this equality with $\phi(z)\phi'(z)$ and then substituting power series expansions of $f(z)$, $\phi(z)$ and $P(z)$ in it, we obtain

$$\begin{aligned}(1-\alpha)(z+2a_2z^2+3a_3z^3+\cdots)&(1+2b_2z+3b_3z^2+\cdots)\\ +\alpha(z+b_2z^2+b_3z^3+\cdots)&(1+4a_2z+9a_3z^2+16a_4z^3+\cdots)\\ =(1+c_1z+c_2z^2+\cdots)&(z+b_2z^2+b_3z^3+\cdots)\\ \cdot(1+2b_2z+3b_3z^2+4b_4z^3+\cdots)&.\end{aligned} \tag{3.7}$$

On comparing the coefficients of z^2, z^3 and z^4 on both sides of (3.7), we obtain

$$2(1+\alpha)a_2 = (1+\alpha)b_2 + c_1, \tag{3.8}$$

$$3(1+2\alpha)a_3 = -4a_2b_2 + (1+2\alpha)b_3 + 3b_2c_1 + 2b_2^2 + c_2, \tag{3.9}$$

and

$$\begin{aligned}4(1+3\alpha)a_4 = &-(6+3\alpha)a_3b_2 + (2\alpha-6)a_2b_3 + (3\alpha-4)b_4\\ &+5b_4 + 5b_2b_3 + (4b_3+2b_2^2)c_1 + 3b_2c_2 + c_3.\end{aligned} \tag{3.10}$$

Since Re $(P(z)) > 0$ for z in E, therefore [6]

$$|c_n| \leqslant 2, \qquad n = 1, 2, 3, \ldots. \tag{3.11}$$

Again $\phi(z)$ being starlike in E, its coefficients satisfy the following inequalities:

$$|b_n| \leqslant n, \qquad n = 2, 3, \ldots, \tag{3.12}$$

and

$$|b_3 - \mu b_2^2| \leqslant 3 - 4\mu, \tag{3.13}$$

if $\mu \leqslant \frac{1}{2}$. (3.12) is well known [6] and (3.13) is due to Keogh and Merkes [2].

On using (3.11) and (3.12), (3.8) gives (3.4). On substituting for a_2 from (3.8) in (3.9), the latter becomes

$$3(1+\alpha)(1+2\alpha)a_3 = (1+3\alpha)b_2c_1 + (1+\alpha)(1+2\alpha)b_3 + (1+\alpha)c_2. \tag{3.14}$$

On using (3.11) and (3.12), we obtain (3.5) from (3.14). Now substitute for a_2 and a_3 from (3.8) and (3.9) respectively in (3.10). Then we arrive at

$$\begin{aligned}(3.15)\qquad 4(1+\alpha)(1+2\alpha)(1+3\alpha)a_4 &= (1+\alpha)(1+2\alpha)(1+3\alpha)b_4\\ &+c_1(1+2\alpha)(1+5\alpha)\left[b_3-\frac{\alpha(1-\alpha)}{(1+2\alpha)(1+5\alpha)}b_2^2\right]\\ &+b_2c_2(1+\alpha)(1+5\alpha)+c_3(1+\alpha)(1+2\alpha).\end{aligned}$$

Notice that $(\alpha(1-\alpha)/((1+2\alpha)(1+5\alpha))) \leqslant 1/2$. Therefore on using (3.11), (3.12) and (3.13) in (3.15), we arrive at

$$\begin{aligned}4(1+\alpha)(1+2\alpha)(1+3\alpha)|a_4| &\leqslant 4(1+\alpha)(1+2\alpha)(1+3\alpha)\\ &+2(1+2\alpha)(1+5\alpha)[3-4\alpha(1-\alpha)/(1+2\alpha)(1+5\alpha)]\\ &+4(1+\alpha)(1+5\alpha)+2(1+\alpha)(1+2\alpha).\end{aligned}$$

On simplification, the above inequality reduces to (3.6).

Equality occurs in (3.4) to (3.6) for $f(z)$ given by (3.3).

It is evident that the above method cannot be easily employed for finding bounds on higher coefficients.

4. **Functions whose derivative has positive real part in the unit disc.** Let $\mathfrak{F}_\alpha$ denote the class of functions $f(z) = z + a_2z^2 + \cdots$ which are regular in E and satisfy the condition

$$(4.1)\qquad \operatorname{Re}(f'(z) + \alpha z f''(z)) > 0$$

for z in E. On putting $\alpha = 0$, we obtain the class $\mathfrak{F}_0$ of functions $f(z)$ whose derivative has positive real part in E.

THEOREM 5. *Let* $f(z) \in \mathfrak{F}_\alpha$ *and* $\operatorname{Re}\alpha \geqslant 0$. *Then* $\operatorname{Re} f'(z) > 0$ *for* z *in* E.

PROOF. The proof follows on choosing $N(z) = zf'(z)$ and $D(z) = z$ in Lemma 1.

It is easy to see that $\mathfrak{F}_\alpha \subset \mathfrak{F}_\beta$ if $\alpha > \beta \geqslant 0$ and that for each fixed $\alpha > 0$ the functions of the class $\mathfrak{F}_\alpha$ are uniformly bounded in E. If $0 \leqslant \lambda \leqslant 1$ and $f(z)$ and $g(z)$ be in $\mathfrak{F}_\alpha$, then $F(z) = (1-\lambda)f(z) + \lambda g(z)$ is also in $\mathfrak{F}_\alpha$. Functions of the class $\mathfrak{F}_\alpha$ are obtained on taking the Hadamard product of the convex function

$$(4.2)\qquad k(z) = \frac{1}{\alpha}z^{1-1/\alpha}\int_0^z \frac{\zeta^{1/\alpha-1}}{1-\zeta}d\zeta$$

with $\int_0^z P(t)\,dt$, where $P(z)$ is an arbitrary function with positive real part in E. (For definition of $f * g$, the Hadamard product of f and g, see [9].)

5. In this section we shall define a class of functions which is closely related to the class $\mathfrak{F}_\alpha$. We say a regular function $f(z)$ is in $\mathcal{G}_\alpha$ if $f(z)$ is regular in E, $f(0) = 0 = f'(0) - 1$, and if it satisfies the condition

$$(5.1)\qquad \operatorname{Re}[(1-\alpha)f(z)/z + \alpha f'(z)] > 0$$

for z in E and for some $\alpha \geqslant 0$. Then $f(z) \in \mathfrak{F}_\alpha$ if and only if $zf'(z) \in \mathcal{G}_\alpha$.

From Lemma 1 it follows that if $f(z) \in \mathcal{G}_\alpha$, then $\operatorname{Re}(f(z)/z) > 0$ for z in

E. On combining this fact with (5.1) we see that $f(z) \in \mathcal{G}_\alpha$ gives $\operatorname{Re}(f'(z)) > 0$ for z in E provided $\alpha \geqslant 1$. Functions of $\mathcal{G}_\alpha$ are univalent in E if $\alpha \geqslant 1$ [8]. MacGregor [3] showed that the functions of the class $\mathcal{G}_0$ are univalent in $|z| < \sqrt{2} - 1$. In Theorem 6 below we shall find sharp radius of univalence of $\mathcal{G}_\alpha$, $0 < \alpha < 1$.

We shall require the following lemma in the proof of the next theorem.

LEMMA 2. *Let $P(z)$ and $Q(z)$ be regular in E, $P(0) = Q(0) = 1$, $\operatorname{Re} P(z) > 0$ and $\operatorname{Re} Q(z) > 1/2$ for z in E. Then $\operatorname{Re}(P(z) * Q(z)) > 0$ for z in E.*

The above lemma is equivalent to a lemma of Nehari and Netanyahu [7]. The following lemma is a simple consequence of the above lemma.

LEMMA 2′. *Let $P(z)$ and $Q(z)$ be regular in E, $P(0) = Q(0) = 1$, $\operatorname{Re} P(z) > 0$ for z in E, and $\operatorname{Re} Q(z) > 1/2$ for z in $|z| < \rho$ $(0 < \rho \leqslant 1)$. Then*

$$\operatorname{Re}(P(z) * Q(z)) > 0$$

for z in $|z| < \rho$.

THEOREM 6. *Let $0 < \alpha < 1$ and $f(z) \in \mathcal{G}_\alpha$. Then $f(z)$ is univalent and satisfies $\operatorname{Re} f'(z) > 0$ in $|z| < r_0$, where r_0 is the smallest positive root of the equation*

$$\frac{2c - 1 - r}{1 + r} - 2c(c - 1)\int_0^1 \frac{t^{c-1}}{1 + tr}\,dt = 0 \tag{5.2}$$

where $c = 1/\alpha$. This result is sharp.

PROOF. Let $f(z) \in \mathcal{G}_\alpha$. Then it is easy to see that there exists a function $P(z)$ which is regular in E and satisfies the conditions $P(0) = 1$ and $\operatorname{Re} P(z) > 0$ for z in E such that

$$f(z) = zP(z) * k(z) \tag{5.3}$$

where $k(z)$ is defined by (4.2). On differentiating both sides of (5.3) with respect to z we obtain

$$f'(z) = P(z) * g(z),$$

where

$$g(z) = k'(z) = \frac{c}{1 - z} - c(c - 1)z^{-c}\int_0^z \frac{\zeta^{c-1}}{1 - \zeta}\,d\zeta. \tag{5.4}$$

Powers in (5.4) are meant as principal values. The function $g(z)$ is regular for z in E, $g(0) = 1$ and

$$2g(z) - 1 = \frac{1 + z}{1 - z} + 2c(c - 1)\int_0^1 t^{c-1}\left[\frac{1}{1 - z} - \frac{1}{1 - tz}\right]dt.$$

It is easy to see that for $|z| = r$, $0 \leqslant r < 1$,

$$2\,\mathrm{Re}\,(g(z)) - 1 \geqslant \frac{1-r}{1+r} - 2c(c-1)\int_0^1 \frac{t^{c-1}(1-t)r}{(1+r)(1+tr)}dt$$

$$= \frac{2c-1-r}{1+r} - 2c(c-1)\int_0^1 \frac{t^{c-1}}{1+tr}dt.$$

Therefore $\mathrm{Re}\,(g(z)) > 1/2$ in $|z| < r_0$ where r_0 is the smallest positive root of (5.2). The theorem now follows from Lemma 2′.

The function

$$f_0(z) = 2k(z) - 1 = z + \sum_{n=2}^{\infty} \frac{2cz^n}{c-1+n}$$

is obtained on putting $P(z) = (1+z)/(1-z)$ in (5.3). Therefore $f_0(z)$ is in $\mathcal{G}_\alpha$. For this function $f_0'(-r_0) = 0$. This shows that the above result is sharp.

The author is grateful to Professor Vikramaditya Singh for his encouragement during the prepration of this paper.

References

1. I. S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc. (2) **3** (1971), 469–474. MR **43** #7611.

2. F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20** (1969), 8–12. MR **38** #1249.

3. T. H. MacGregor, *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc. **104** (1962), 532–537. MR **25** #4090.

4. P. T. Mocanu, *Une propriété de convexité généralisée dans la théorie de la représentation conforme*, Mathematica (Cluj) **11 (34)** (1969), 127–133. MR **42** #7881.

5. S. S. Miller, P. T. Mocanu and M. O. Reade, *All α-convex functions are univalent and starlike*, Proc. Amer. Math. Soc. **37** (1973), 553–554. MR **47** #2044.

6. Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952. MR **13**, 640.

7. Z. Nehari and E. Netanyahu, *On the coefficients of meromorphic schlicht functions*, Proc. Amer. Math. Soc. **8** (1957), 15–23. MR **18**, 648.

8. K. Noshiro, *On the theory of schlicht functions*, J. Fac. Sci. Hokkaido Univ. (1) **2** (1934/35), 129–155.

9. St. Ruscheweyh and T. Sheil-Small, *Hadamard products of schlicht functions and the Pólya-Schoenberg conjecture*, Comment. Math. Helv **48** (1973), 119–135. MR **48** #6393.

10. K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan **11** (1959), 72–75. MR **21** #5734.

Department of Physics, Punjabi University, Patiala-147002, India