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ON CURVILINEAR CLUSTER SETS ON OPEN RIEMANN SURFACES

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ABSTRACT. Every boundary point of the Kerékjártó-Stoïlow compactification of an open Riemann surface is the limit of a Jordan arc with this property: for every nonempty continuum in the extended complex plane there is a holomorphic function on the surface having the continuum as its cluster set along the arc.

Let R be an open Riemann surface and $\{R_n\}_{n=1}^{\infty}$ a regular exhaustion of R. Let R^* denote the Kerékjártó-Stoïlow compactification of R and $\{G_n\}_{n=1}^{\infty}$ a determinant sequence of $e \in \Delta$ satisfying $\partial G_n \subset \partial R_n$, where $\Delta = R^* - R$ and ∂X means the relative boundary of $X \subset R$ with respect to R.

The approximation theorem of Bishop (cf. [2]), which is applied in the proof of our Theorem, is stated as follows:

Let R' be an open Riemann surface and K' a compact subset with the property that no nonempty component of $R' - K'$ is relatively compact. Let g' be a continuous function on K' which is holomorphic at interior points. Then for any $\varepsilon' > 0$ there exists a holomorphic function f' on R' for which $|f' - g'| < \varepsilon'$ on K'.

In this paper, we shall show the following:

THEOREM. *For each $e \in \Delta$ and any nonempty continuum K in the Riemann sphere S, there exist a holomorphic function f on R and a Jordan arc γ in R converging to e such that $C_\gamma(f, e) = K$, where $C_\gamma(f, e)$ denotes the cluster set of f on γ.*

PROOF. Fix the chordal metric d on S. Let U_n be the $1/n$-neighborhood of K relative to d. Evidently U_n is both open and connected. Let $0 = t_0 < t_1 < \cdots < t_n < \cdots$, $t_n \to 1$, and for $n \geq 1$ let $g_n\colon [t_{n-1}, t_n] \to U_n - \{\infty\}$ be continuous such that

(1) no point of K is at a d-distance of more than $1/n$ from the image of g_n,

(2) for $n \geq 2$, $g_n(t_{n-1}) = g_{n-1}(t_{n-1})$.

Then $\bigcup g_n$ defines a continuous $g\colon [0, 1) \to S$ such that $\bigcap_{t<1} \overline{g([t, 1))} = K$, where the bar implies closure.

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It is easy to see that there is an arc γ defined by a topological h: $[0, 1) \to R$ such that $h(t) \to e$ as $t \to 1$, $h([t_{n-1}, t_n]) \subset \overline{R}_{n+1} - R_n$, and $\gamma \cap \partial G_n = \{h(t_{n-1})\}$. Define $\varphi = g \circ h^{-1}$ on $\gamma \subset R$ and extend φ by Tietze's theorem to a continuous complex-valued function φ on R. This is possible since g is never ∞.

We may assume, without loss of generality, that $\varphi = 0$ on $\overline{R}_1$. Set $\varphi_1 \equiv 0$ on R and $\psi_2 = \varphi - \varphi_1$ on $\overline{R}_2$; then from the approximation theorem of Bishop, there exists a holomorphic function f_2 on R for which $|f_2 - \psi_2| < 2^{-3}$ on $\overline{R}_1 \cup h([0, t_1])$.

Next let $\lambda(t)$ $(t_1 \leqslant t < 1)$ be a continuous function with the property that $\lambda(t_1) = \psi_2 \circ h(t_1)$, $\lambda(t_2) = f_2 \circ h(t_2)$, $|f_2 \circ h - \lambda| < 2^{-2}$ on $[t_1, t_2]$ and $\lambda = f_2 \circ h$ on $[t_2, 1)$. Let φ_2 be a continuous function on R such that $\varphi_2 = \psi_2$ on $\overline{R}_2$ and $\varphi_2 = \lambda \circ h^{-1}$ on $(R - R_2) \cap \gamma$. Then we see that $|f_2 - \varphi_2| < 2^{-2}$ on $\overline{R}_1 \cup \gamma$, $\varphi_2 = f_2$ on $(R - R_3) \cap \gamma$ and $\varphi_2 = \varphi - \varphi_1$ on $\overline{R}_2$.

By mathematical induction, we have a sequence $\{f_n\}_{n=2}^{\infty}$ of holomorphic functions on R and a sequence $\{\varphi_n\}_{n=2}^{\infty}$ of continuous functions on R with the property that $|f_n - \varphi_n| < 2^{-n}$ on $\overline{R}_{n-1} \cup \gamma$, $\varphi_n = f_n$ on $(R - R_{n+1}) \cap \gamma$ and $\varphi_n = \varphi - (\varphi_1 + \varphi_2 + \cdots + \varphi_{n-1})$ on $\overline{R}_n$. Since $\varphi_n = 0$ on $\overline{R}_{n-1}$, $\sum_{n=2}^{\infty} \varphi_n$ converges on R and $\varphi = \sum_{n=2}^{\infty} \varphi_n$ on R. Since $|f_n| < 2^{-n}$ on $\overline{R}_{n-1}$, $\sum_{n=2}^{\infty} f_n$ converges uniformly on every compact subset of R, and hence $f = \sum_{n=2}^{\infty} f_n$ is holomorphic on R.

Now for any $\varepsilon > 0$, there exists an $N = N(\varepsilon)$ such that $\sum_{n=N}^{\infty} |f_n - \varphi_n| < \varepsilon$ on γ. Further since $\sum_{n=2}^{N-1} |f_n - \varphi_n| = 0$ on $(R - R_N) \cap \gamma$, we have $|f - \varphi| < \varepsilon$ on $(R - R_N) \cap \gamma$. Since $f - \varphi \to 0$ along γ, we see that $f(h(t)) - g(t) = f(h(t)) - \varphi(h(t)) \to 0$ as $t \to 1$. Clearly, the cluster set of f along γ is the cluster set of g along $[0, 1)$, which we have seen is K.

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