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POLYNOMIAL DENSITY IN BERS SPACES

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ABSTRACT. Let D be a bounded Jordan domain such that $\int\int_D \lambda_D^{2-q}\,dx\,dy < \infty$ for $q > 1$. Here $\lambda_D(z)$ is the Poincaré metric for D. Define $A_q^p(D)$, the Bers space, to be the Fréchet space of holomorphic functions f on D, such that $\|f\|_{q,p}^p = \int\int_D \lambda_D^{2-qp}|f|^p\,dx\,dy$ is finite, $0 < p < \infty$, $qp > 1$. It is well known that the polynomials are dense in $A_q^p(D)$ for $qp \geq 2$. We show that they are dense in $A_q^p(D)$ for $qp > 1$ irrespective whether the boundary of D is rectifiable or not.

1. **Introduction.** Let D be a bounded Jordan domain such that

$$\iint_D \lambda_D^{2-q}(z)\,dx\,dy < \infty \tag{1.1}$$

where $\lambda_D(z)$ is the Poincaré metric for D. Since $\lambda_D^{-1}(z) \leqslant \sqrt{A/\pi}$, where A is the area of D, it follows that (1.1) obviously holds for all $q \geqslant 2$. Hence we can assume that (1.1) holds for all $q > q_0$ where $1 \leqslant q_0 < 2$ (the case $q \leqslant 1$ is, of course, trivial). Let ϕ: $D \to U$ be a Riemann mapping of D onto U, the unit disc, and let $\psi = \phi^{-1}$. Then

$$\iint_D \lambda_D^{2-q}(z)\,dx\,dy = \iint_U \left(1 - |w|^2\right)^{q-2}|\psi'(w)|^q\,du\,dv. \tag{1.2}$$

A well-known inequality due to Hardy and Littlewood [5] states that for $q > 1$, $r > 0$,

$$\left\{\iint_U \left(1 - |w|^2\right)^{q-2}|f(w)|^r\,du\,dv\right\}^{1/r} \leqslant c\|f\|_{r/q}, \tag{1.3}$$

where c is a constant depending on q and r and $\|\;\|_{r/q}$ is the $H^{r/q}$ norm. Here $H^{r/q} = H^{r/q}(U)$ is the r/q Hardy class.

Since for Jordan domains the rectifiability of the boundary is equivalent to $\psi' \in H^1(U)$ [3, p. 44], it follows from (1.2) and (1.3) that

PROPOSITION 1. *Let D be a bounded rectifiable Jordan domain. Then* (1.1) *holds for all* $q > 1$.

However, the property that (1.1) holds for all $q > 1$ is not characteristic to rectifiable domains. In fact,

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PROPOSITION 2. *There is a domain D, bounded by a nonrectifiable Jordan curve, such that* (1.1) *holds for all* $q > 1$.

PROOF. According to Hedberg [6] there is such a domain D with

$$\iint_D \left(1 - |\phi(z)|^2\right)^{q-2} dx\, dy < \infty \tag{1.4}$$

for all $q > 1$. Since (1.1) holds for all $q \geqslant 2$ we can assume that $1 < q < 2$. An application of Hölder's inequality yields

$$\iint_U \left(1 - |w|^2\right)^{q-2} |\psi'(w)|^q\, du\, dv$$
$$\leqslant \left\{ \iint_U \left(1 - |w|^2\right)^{q-2} |\psi'(w)|^2\, du\, dv \right\}^{q/2} \cdot \left(\frac{\pi}{q-1} \right)^{1-q/2}.$$

The proposition now follows from (1.4).

We shall only consider those domains D such that (1.1) holds for all $q > 1$, those D for which (1.1) holds for $q > q_0 > 1$ will be considered elsewhere. For $0 < p < \infty$ and $qp > 1$ we define $A_q^p(D)$, the Bers space, as the Fréchet space of holomorphic functions $f(z)$ on D, "normed" by

$$\|f\|_{q,p} = \left\{ \iint_D \lambda_D^{2-qp}(z) |f(z)|^p\, dx\, dy \right\}^{1/p}.$$

Clearly $A_q^p(D)$ is a Banach space for $1 \leqslant p < \infty$, $qp > 1$, and it is a Fréchet space for $0 < p < 1$, $qp > 1$, with the usual metric $d(f, g) = \|f - g\|_{q,p}^p$, $f, g \in A_q^p(D)$. Also, since D is bounded, the assumption about (1.1) implies that the polynomials belong to $A_q^p(D)$ for all $0 < p < \infty$ and $qp > 1$.

The question of polynomial density in $A_q^1(D)$ has been considered by various authors. For $q \geqslant 2$, Bers [2] and Knopp [7] proved that the polynomials are dense in $A_q^1(D)$ without any assumption on the mapping function ψ. Later Sheingorn [10] proved that the polynomials are dense in $A_q^1(U^*)$, $1 < q < \infty$, where U^* is a special Jordan domain introduced first by Earle and Marden [4] and used by Knopp [7] to prove his main lemma. Metzger [8] proved that if $\psi' \in H^1(U)$ and $q > \frac{3}{2}$ then the polynomials are dense in $A_q^1(D)$. Recently, Metzger [9] was able to improve his result, and he actually showed that if $\psi' \in H^1(U)$ then the polynomials are dense in $A_q^1(D)$ for all $q > 1$. Our contribution in this paper is in showing that the polynomials are dense in $A_q^1(D)$ for all $q > 1$ without any assumption on the boundary behavior of ψ', and, in view of Propositions 1 and 2, Metzger's results are obtained as a special case. In fact we will prove

THEOREM 1. *Let D be a bounded Jordan domain. Then the polynomials are dense in $A_q^p(D)$ for* $0 < p < \infty$, $qp > 1$.

In order to prove this theorem we consider the space $\mathcal{H}_q^p(D) = A_{q/p}^p(D)$ instead of A_q^p. Therefore, $\mathcal{H}_q^p(D)$ is the Fréchet space of holomorphic functions $f(z)$ on D normed by

$$\|f\|_{q,p} = \left\{ \iint_D \lambda_D^{2-q}(z)|f(z)|^p \, dx \, dy \right\}^{1/p},$$

where $q > 1$, $0 < p < \infty$. Here $\|1\|_{q,1} = \iint_D \lambda_D^{2-q}(z) \, dx \, dy < \infty$ for all $q > 1$.

Using this notation, Theorem 1 can be restated as follows:

THEOREM 1′. *Let D be a bounded Jordan domain. Then the polynomials are dense in $\mathcal{H}_q^p(D)$ for $0 < p < \infty$, $q > 1$.*

2. **Auxiliary facts.** In the case $q \geqslant 2$, Theorem 1′ was actually proved by Bers [2] although his result is stated for only the case $p = 1$.

LEMMA 1 (BERS). *Let D be a bounded Jordan domain. Then the polynomials are dense in $\mathcal{H}_q^p(D)$ for $0 < p < \infty$, $q \geqslant 2$.*

The following lemma is by now standard.

LEMMA 2. *The polynomials are dense in $\mathcal{H}_q^p(U)$ for $0 < p < \infty$, $q > 1$.*

Using the Carathéodory-Walsh theorem [11, p. 36] we can show (see also [8], [10])

LEMMA 3. *Let $0 < p < \infty$, $q > 1$. The polynomials are dense in $\mathcal{H}_q^p(D)$ if and only if $(\phi')^{q/p}$ is in the $\mathcal{H}_q^p(D)$-closure of the polynomials.*

The following technical lemma is needed for proving the main theorem.

LEMMA 4. *Let $\alpha > 0$ and $1 < s < \infty$ such that*

$$(1 - 1/\alpha)s = 1 - Q/2, \qquad Q > 1. \tag{2.1}$$

If

$$s(1 + q - 2/\alpha) > 1, \tag{2.2}$$

then

$$\mathcal{H}_Q^{ps}(D) \subset \mathcal{H}_q^p(D), \qquad q > 1, \tag{2.3}$$

the injection being continuous. If also

$$s[q + 4(1 - 1/\alpha)] < 3, \tag{2.4}$$

then $(\phi')^{q/p}$ is in $\mathcal{H}_Q^{ps}(D)$.

PROOF. We have

$$\|f\|_{q,p}^p = \iint_D \lambda_D^{2-2/\alpha+2/\alpha-q}|f|^p \, dx \, dy.$$

An application of Hölder's inequality with $1 < s < \infty$ and $s' = s/(s-1)$ yields

$$\|f\|_{q,p}^p \leqslant \left\{ \iint_D \lambda_D^{2(1-1/\alpha)s}|f|^{ps} \, dx \, dy \right\}^{1/s}$$
$$\times \left\{ \iint_D \lambda_D^{2(1/\alpha - q/2)s/(s-1)} \, dx \, dy \right\}^{(s-1)/s}$$

The first integral on the right-hand side is the $\mathcal{H}_Q^{ps}(D)$ norm, as (2.1) shows. The second integral is finite if $2(1/\alpha - q/2)s/(s-1) < 1$ which is exactly (2.2). Therefore, (2.3) is proved. We now show that $(\phi')^{q/p}$ is in $\mathcal{H}_Q^{ps}(D)$ under the above conditions. We, of course, can assume that $0 \in D$ and that $\phi(0) = 0$. Now,

$$\|(\phi')^{q/p}\|_{Q,ps}^{ps} = \iint_D \lambda_D^{2(1-1/\alpha)s} |\phi'(z)|^{qs}\, dx\, dy$$
$$= \iint_U \left(1 - |w|^2\right)^{2(1/\alpha - 1)s} |\psi'(w)|^{2-s-s(1+q-2/\alpha)}\, du\, dv.$$

Since $1 < s < \infty$, (2.2) implies that the exponent of $|\psi'(w)|$ in the above integral is negative. Also, since ψ is a bounded schlicht function with $\psi(0) = 0$, it follows that $|\psi'(w)| \geqslant M(1 - |w|^2)$ for all $w \in U$ for some positive constant M. Therefore

$$\|(\phi')^{q/p}\|_{Q,ps}^{ps} \leqslant M_1 \iint_U \left(1 - |w|^2\right)^{4s/\alpha - 4s - sq + 2} du\, dv,$$

for another positive constant M_1. The last integral is finite if and only if (2.4) holds. This concludes the proof of the lemma. Note, however, that conditions (2.1), (2.2), and (2.4) are independent of p $(0 < p < \infty)$.

3. **Proof of Theorem** 1′. The idea of the proof is to perturb α and s, subject to the restrictions of Lemma 4, so that we have polynomial approximation in $\mathcal{H}_Q^{ps}(D)$, and it suffices, according to Lemma 3, to show that $(\phi')^{q/p}$ is in $\mathcal{H}_Q^{ps}(D)$. The proof is done by successive perturbations. The result of Metzger [8] will be obtained as a special case of the first perturbation. Corresponding to Lemma 4 we let

$$\alpha_n = 2\frac{n+2}{n+4}, \qquad n = 0, 1, \ldots;$$

$$Q_0 = 2, \quad Q_n > 1 + \frac{1}{n+1}, \qquad n = 1, 2, \ldots,$$

and

$$s_n > 1, \quad (1 - 1/\alpha_n)s_n = 1 - Q_n/2, \qquad n = 0, 1, \ldots.$$

Note that s_0 is free except, of course, that $s_0 > 1$. We now proceed by induction on n to show that polynomials are dense in $\mathcal{H}_q^p(D)$ for all $0 < p < \infty$ and all $q > 1 + 1/(n+2)$.

For $n = 0$, $\alpha_0 = 1$, $Q_0 = 2$ and $s_0 > 1$. Using Lemma 1 we can assume that $1 < q < 2$. According to Lemma 1 the polynomials are dense in $\mathcal{H}_{Q_0}^{ps_0}(D) = \mathcal{H}_2^{ps_0}(D)$. Using Lemma 4, $(\phi')^{q/p}$ is in $\mathcal{H}_2^{ps_0}(D)$ if $s_0(q-1) > 1$ and $s_0 q < 3$, that is, if $1/(q-1) < s_0 < 3/q$. Our assumption $1 < q < 2$ guarantees the existence of such $s_0 > 1$. Therefore $(\phi')^{q/p} \in \mathcal{H}_2^{ps_0}(D)$ if $q > 3/2$ and it follows by Lemmas 3 and 4 that the polynomials are dense in $\mathcal{H}_q^p(D)$ for all $0 < p < \infty$ and $q > 3/2$. Assume now we have proved that the polynomials are dense in $\mathcal{H}_q^p(D)$ for all $0 < p < \infty$ and all $q > 1 + 1/(k+1)$, $k =$

$1, 2, \ldots, n$. We shall show that this is true also for $q > 1 + 1/(n+2)$. Then we can assume that $1 < q < 2/\alpha_n$ because $2/\alpha_n = 1 + 2/(n+2) > 1 + 1/(n+1)$. Since $Q_n > 1 + 1/(n+1)$ it follows from the induction hypothesis that the polynomials are dense in $\mathcal{H}^{ps_n}_{Q_n}(D)$. By Lemma 4, $(\phi')^{q/p}$ is in $\mathcal{H}^{ps_n}_{Q_n}(D)$ if $s_n(1 + q - 2/\alpha_n) > 1$ and $s_n[q + 4(1 - 1/\alpha_n)] < 3$; that is, if

$$\frac{1}{1 + q - 2/\alpha_n} < s_n < \frac{3}{q + 4(1 - 1/\alpha_n)}, \tag{3.1}$$

and our choice of q $(1 < q < 2/\alpha_n)$ shows that (3.1) is contained in the range of $1 < s_n < (n+2)/(n+1)$ if and only if $q > 1 + 1/(n+2)$ and then by Lemmas 3 and 4 the theorem follows.

4. **Concluding remarks.** We first note that Theorem 1 has the following

COROLLARY 1. *Let G be a Fuchsian group acting on D. Then the set of Poincaré series of polynomials is dense in $A^1_q(D, G)$, $q > 1$ (cf. Bers* [1] *and Knopp* [7] *for the appropriate formulation*).

If we introduce the class $E^p(D)$ as, for example, in Duren [3, p. 168], then $E^p(U) = H^p(U)$, $0 < p < \infty$. If D is a bounded Jordan domain then $E^p(D)$ is a Fréchet space of holomorphic functions on D normed by

$$\|f\|^p_{p,D} = \operatorname*{Sup}_{r<1} \int_{\gamma_r} |f(z)|^p |dz|$$

where γ_r is the image under ψ of the circle $|w| = r$. Since $A^p_q(D)$ and $E^{1/q}(D)$ are preserved under the same isometry induced by conformal mappings, it follows immediately from (1.3) that $E^{1/q}(D) \subset A^p_q(D)$, $0 < p < \infty$, $qp > 1$. Using Theorem 1, we obtain

COROLLARY 2. *$E^{1/q}(D)$ is dense in $A^p_q(D)$, $0 < p < \infty$, $qp > 1$.*

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