PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 62, Number 1, January 1977

MONOTONE AND OSCILLATORY SOLUTIONS OF $y^{(n)} + py = 0$

W. J. KIM

ABSTRACT. Monotone and oscillatory behaviors of the solutions with the property that $y(x)/x^2 \to 0$ as $x \to \infty$ or $y(x)/x \to 0$ as $x \to \infty$ are discussed. For example, it is shown that every nonoscillatory solution y, such that $y(x)/x \to 0$ as $x \to \infty$, monotonically tends to zero as $x \to \infty$, provided n is odd, $p \geqq 0$, and $\int^\infty x^{n-1}p(x)\,dx = \infty$.

The differential equation to be considered is

$$y^{(n)} + py = 0, \tag{E}$$

where p is continuous and of one sign on an interval $[a, \infty)$. Since the behaviors of solutions of (E) depend on the parity of n and the sign of p, it is natural to classify (E) into the following four cases:

(i) $\quad n$ even, $p \geqq 0$,

(ii) $\quad n$ odd, $p \geqq 0$,

(iii) $\quad n$ even, $p \leqq 0$,

(iv) $\quad n$ odd, $p \leqq 0$.

In the sequel, (E_i), for example, shall denote the equation (E) satisfying condition (i). Likewise, (E_{ii}), (E_{iii}), and (E_{iv}) denote equation (E) satisfying (ii), (iii), and (iv), respectively.

A nontrivial solution of (E) is said to be *oscillatory* on $[a, \infty)$ if it has an infinity of zeros on $[a, \infty)$; otherwise, it is said to be *nonoscillatory* on $[a, \infty)$. If every solution of (E) is nonoscillatory on $[a, \infty)$, (E) is said to be *nonoscillatory* on $[a, \infty)$. On the other hand, if (E) has an oscillatory solution, (E) is said to be *oscillatory*.

Oscillatory and nonoscillatory behaviors of (E) have been extensively studied by a number of authors [1]–[13]. In particular, Anan'eva and Balaganski [1] proved the following statements: If $p > 0$ and $\int_a^\infty x^{n-2}p(x)\,dx = \infty$, then every solution of (E) is oscillatory if n is even, while every nonoscillatory

Received by the editors September 22, 1975.
AMS (*MOS*) *subject classifications* (1970). Primary 34C10.
Key words and phrases. Monotone and oscillatory solutions, asymptotic behavior, linear equations, ordinary, nth-order, real-valued continuous coefficients

solution of (E) with odd n satisfies

$$\lim_{x\to\infty} y^{(n-1)}(x) = \cdots = \lim_{x\to\infty} y'(x) = \lim_{x\to\infty} y(x) = 0,$$

where the signs of $y, y', \ldots, y^{(n)}$ are preserved for sufficiently large x and successively alternating. Similar results were obtained by Kondrat'ev [4] under different conditions imposed on the coefficient p. Central to the proofs of these results is that any solution y with $y(x) \to 0$ as $x \to \infty$ is oscillatory, provided equation (E) meets certain requirements.

It is our aim to improve and extend some aspects of the above results. For example, it will be shown that every nontrivial solution y of (E_i) or (E_{iv}) such that $y(x)/x^2 \to 0$ as $x \to \infty$, either is oscillatory or satisfies a sequence of inequalities, which has interesting consequences. We shall also prove that every nonoscillatory solution y of (E_{ii}) [(E_{iii})] such that $y(x)/x \to 0$ as $x \to \infty$ tends to zero as $x \to \infty$ if $\int_a^\infty x^{n-1}p(x)\,dx = \infty$ [$-\infty$].

THEOREM 1. *Assume that $p \not\equiv 0$ on $[a_1, \infty)$ for every $a_1 \geqq a$. If y is a nontrivial solution of* (E_i) *or* (E_{iv}) *such that $y \geqq 0$ and $y(x)/x^2 \to 0$ as $x \to \infty$, then*

$$(1)\qquad \begin{aligned} &y(x) \geqq 0, \quad y'(x) > 0, \quad y''(x) < 0, \\ &\qquad y'''(x) > 0, \ldots, (-1)^n y^{(n-1)}(x) > 0, \qquad x \in [a, \infty), \end{aligned}$$

and $y^{(k)}(x) \to 0$ monotonically as $x \to \infty$, $k = 2, 3, \ldots, n-1$.

PROOF. Put $y_1 = y, y_2 = y', \ldots, y_n = y^{(n-1)}$ and let b be an arbitrary point of $[a, \infty)$. Then y satisfies the system

$$y_1(x) = y_1(b) + \int_b^x y_2(t)\,dt,$$

$$\cdots$$

$$y_{n-1}(x) = y_{n-1}(b) + \int_b^x y_n(t)\,dt,$$

$$y_n(x) = y_n(b) - \int_b^x p(t)y_1(t)\,dt.$$

Suppose $y = y_1$ is a solution of (E_i). Then $\int_b^x p(t)y_1(t)\,dt$ is a nondecreasing, nonnegative function of x and clearly is positive on an interval $[c, \infty)$ for some $c > b$. We claim that $y_n(b) > 0$. To prove this, assume the contrary: $y_n(b) \leqq 0$. Then, $y_n(x)$ is nonpositive, nonincreasing on $[b, \infty)$ and

$$y_n(c) = y_n(b) - \int_b^c p(t)y_1(t)\,dt < 0,$$

that is, $y_n(x) \leqq y_n(c) < 0$, $x \in [c, \infty)$. Consequently, $y_{n-1}(x) \to -\infty$ as $x \to \infty$, irrespective of $y_{n-1}(b)$. This in turn implies $y_{n-2}(x) \to -\infty$ as $x \to \infty$, and successively $y_k(x) \to -\infty$ as $x \to \infty$, regardless of the value $y_k(b)$, $k = 1, 2, \ldots, n-1$. In particular, $y_1(x) = y(x) \to -\infty$ as $x \to \infty$, contrary to the hypothesis that $y \geqq 0$ on $[a, \infty)$. This contradiction proves $y_n(b) > 0$. Since b is arbitrary, we conclude that $y_n(x) > 0$, $x \in [a, \infty)$. It is now easy to see that $y_n(x) \to 0$ as $x \to \infty$ for $n > 2$. If this were not the case, there would exist a constant $K > 0$ such that $y_n(x) > K$, $x \in [c_1, \infty)$, for some $c_1 \geqq a$. However, this implies that $y(x) > K_1 x^{n-1}$ on $[c_2, \infty)$ for some constants $K_1 > 0$ and $c_2 > c_1$, contradicting the asymptotic behavior $y(x)/x^2 \to 0$ as $x \to \infty$. Next, we shall prove that $y_{n-1}(x) < 0$ if $n > 2$. Evidently, $y_{n-1}(x)$ is a monotonically increasing function. If $y_{n-1}(b) \geqslant 0$, then $y_{n-1}(x) \geqq 0$ on $[b, \infty)$, and there would exist constants $C > 0$ and $d > b$ such that $y_{n-1}(x) > C$, $x \in [d, \infty)$. However, for $n > 2$, this again leads to the contradiction that $y(x) > Cx^{n-2}$, $x \in [d_1, \infty)$, for some $d_1 > d$. Thus, $y_{n-1}(b) < 0$, and $y_{n-1}(x) < 0$ since b is arbitrary. Moreover, we must have $y_{n-1}(x) \to 0$ as $x \to \infty$, for otherwise we would again be led to the contradiction that $y(x) \to -\infty$ as $x \to \infty$. In this way, we can successively establish the inequalities

$$y_n(x) > 0, \quad y_{n-1}(x) < 0, \ldots, y_4(x) > 0, \quad y_3(x) < 0, \qquad x \in [a, \infty),$$

with the property that $y_k(x) \to 0$ as $x \to \infty$, $k = 3, 4, \ldots, n$. Continuing this process, we deduce $y_2(x) > 0$ and $y_1(x) \geqq 0$, $x \in [a, \infty)$. This proves the theorem for $(\mathrm{E_i})$. The proof for $(\mathrm{E_{iv}})$ is similar; in this case, we first prove that $y_n(x) < 0$ and $y_n(x) \to 0$ as $x \to \infty$, and continue as in the case of $(\mathrm{E_i})$.

In a somewhat similar fashion, we can prove

THEOREM 2. *Assume that $p \not\equiv 0$ on $[a_1, \infty)$ for every $a_1 \geqq a$. If y is a nontrivial solution of $(\mathrm{E_{ii}})$ or $(\mathrm{E_{iii}})$ such that $y \geqq 0$ on $[a, \infty)$ and $y(x)/x \to 0$ as $x \to \infty$, then*

$$y(x) > 0, \quad y'(x) < 0, \quad y''(x) > 0, \ldots, (-1)^n y^{(n-1)}(x) < 0, \qquad x \in [a, \infty), \tag{2}$$

and $y^{(k)}(x) \to 0$ monotonically as $x \to \infty$, $k = 1, 2, \ldots, n - 1$.

In order to characterize the behaviors of solutions, we may reformulate Theorem 1 as follows:

COROLLARY 1. *Suppose $p \not\equiv 0$ on $[a_1, \infty)$ for every $a_1 \geqq a$. Let y be a nontrivial solution of $(\mathrm{E_i})$ or $(\mathrm{E_{iv}})$ such that $y(x)/x^2 \to 0$ as $x \to \infty$. Then either*

(a) *y is oscillatory on $[a, \infty)$, or else*

(b) *$y \geqq 0$ [$\leqq 0$] on $[b, \infty)$, for some $b \geqq a$, and y [$-y$] satisfies the inequalities in (1) of Theorem 1. In particular, y [$-y$] increases [decreases] monotonically on $[b, \infty)$.*

If y is a nontrivial solution of $(\mathrm{E_i})$ or $(\mathrm{E_{iv}})$ such that $y(x) \to 0$ as $x \to \infty$, it cannot satisfy the inequalities in (1) of Theorem 1. Thus, we conclude by Corollary 1 that y is oscillatory.

THEOREM 3. *If $\int_a^\infty x^{n-1} p(x)\, dx = \infty$ [$-\infty$], every nonoscillatory solution of $(\mathrm{E_i})$ [$(\mathrm{E_{iv}})$] is unbounded on $[a, \infty)$.*

PROOF. Assume the contrary. Suppose there exists a nontrivial solution y which is bounded and positive on $[b, \infty)$, for some $b \geqq a$. Since y increases monotonically by Theorem 1, there exist two positive constants M_1 and M_2 such that $M_1 \leqq y(x) \leqq M_2$, $x \in [b, \infty)$. Multiplying (E) by x^{n-1} and integrating n times by parts, we get

$$
\begin{aligned}
(3)\qquad & x^{n-1}y^{(n-1)} - (n-1)x^{n-2}y^{(n-2)} + (n-1)(n-2)x^{n-3}y^{(n-3)} \\
& \qquad - \cdots + (-1)^{n-1}(n-1)!\,y = -\int_b^x t^{n-1}p(t)y(t)\,dt + C,
\end{aligned}
$$

where C is a constant.

If y is a solution of (E_i), then n is even, $p \geqq 0$, and we get from (3),

$$
\begin{aligned}
(4)\qquad & x^{n-1}y^{(n-1)} - (n-1)x^{n-2}y^{(n-2)} + \cdots \\
& \qquad + [(n-1)(n-2)\cdots 2]xy' - (n-1)!\,M_2 \\
& \qquad < -M_1 \int_b^x t^{n-1}p(t)\,dt + C, \qquad x \in [b, \infty).
\end{aligned}
$$

On the other hand, since $y^{(n-1)} > 0$, $y^{(n-2)} < 0, \ldots, y' > 0$ by Theorem 1, the left-hand side of (4) cannot tend to $-\infty$ as $x \to \infty$, while the right-hand side tends to $-\infty$ as $x \to \infty$. Therefore, inequality (4) cannot hold throughout $[b, \infty)$. This incompatibility proves that the solution y must be unbounded on $[a, \infty)$.

If y is a solution of (E_{iv}), then n is odd, $p \leqq 0$, and we get from (3),

$$
\begin{aligned}
(5)\qquad & x^{n-1}y^{(n-1)} - (n-1)x^{n-2}y^{(n-2)} + \cdots \\
& \qquad -[(n-1)(n-2)\cdots 2]xy' + (n-1)!\,M_2 \\
& \qquad > -M_1 \int_b^x t^{n-1}p(t)\,dt + C, \qquad x \in [b, \infty).
\end{aligned}
$$

In this case, $y^{(n-1)} < 0$, $y^{(n-2)} > 0, \ldots, y' > 0$, due to Theorem 1. For this reason, the left-hand side of (5) cannot approach ∞ as $x \to \infty$, while the right-hand side approaches ∞ as $x \to \infty$. Hence, inequality (5) cannot hold throughout $[b, \infty)$, and our assumption that y is bounded must be false.

Theorem 3 may be restated as follows: If (E_i) $[(E_{iv})]$ has a bounded nonoscillatory solution on $[a, \infty)$, then $\int_a^\infty x^{n-1}p(x)\,dx < \infty$ $[> -\infty]$. However, the above inequality is known to guarantee the nonoscillation of (E_i) $[(E_{iv})]$ on $[a, \infty)$ [4], [11]. Thus, we have the following result.

COROLLARY 2. *If* (E_i) $[(E_{iv})]$ *has a bounded nonoscillatory solution defined on* $[a, \infty)$, *then* (E_i) $[(E_{iv})]$ *is nonoscillatory on* $[a, \infty)$.

According to a result of Kondrat'ev [4], any solution y of $y^{(n)} + py = 0$, $p \in C[a, \infty)$, $p \geqq 0$, such that $y(x) \to 0$ as $x \to \infty$ and $y(b) = 0$, for some $b \in [a, \infty)$, is oscillatory on $[a, \infty)$. In this connection, we have the following corollary of Theorem 2.

COROLLARY 3. *Suppose* $p \not\equiv 0$ *on* $[a_1, \infty)$ *for every* $a_1 \geqq a$. *Let* y *be a nontrivial solution of* (E_{ii}) *or* (E_{iii}) *such that* $y(x)/x \to 0$ *as* $x \to \infty$. *If* y *violates any of the inequalities in* (2) *of Theorem* 2, *then it is oscillatory on* $[a, \infty)$. *In particular, if* $y^{(k)}(b) = 0$ *for some* k, $0 \leqq k \leqq n - 1$, *and some point* $b \in [a, \infty)$, *then* y *is oscillatory on* $[a, \infty)$.

It is well known that equation (E_{ii}) and (E_{iii}) have a nontrivial solution y satisfying the inequalities

$$(6) \qquad (-1)^j y^{(j)}(x) \geqq 0, \qquad j = 0, 1, \ldots, n-1, \quad x \in [a, \infty),$$

which is due to Hartman and Wintner [2]. In a recent paper [9, Theorem 4], Read showed that any solution $z = z(x)$ of (E_{iii}) satisfying (6), tends to zero as $x \to \infty$ if and only if $\int_a^\infty t^{n-1} p(t)\,dt = -\infty$. We shall establish a similar result for (E_{ii}).

LEMMA. *Suppose* y *is a nontrivial solution of* (E_{ii}) *which satisfies* (6). *Then* $y(x) \to 0$ *as* $x \to \infty$ *if* $\int_a^\infty x^{n-1} p(x)\,dx = \infty$.

PROOF. Suppose $y(x)$ does not approach 0 as $x \to \infty$. Then there exists a positive constant M such that $y(x) > M$, $x \in [a, \infty)$. Using (3) with b replaced by a, and recalling that n is odd and $p \geqq 0$, we deduce

$$\begin{aligned} x^{n-1} y^{(n-1)} - (n-1) x^{n-2} y^{(n-2)} + \cdots \\ - [(n-1)(n-2)\cdots 2] x y' + (n-1)!\, M \\ < -M \int_a^x t^{n-1} p(t)\,dt + C_1, \qquad x \in [a, \infty), \end{aligned}$$

where C_1 is a constant. Since the left-hand side of this inequality is positive on $[a, \infty)$ because of (6), $\int_a^x t^{n-1} p(t)\,dt$ cannot tend to ∞ as $x \to \infty$. This proves the Lemma.

In view of Theorem 2, Theorem 4 of Read [9], and the Lemma, we have

THEOREM 4. *If* $\int_a^\infty x^{n-1} p(x)\,dx = \infty\ [-\infty]$, *every nonoscillatory solution* y *of* (E_{ii}) $[(E_{iii})]$ *such that* $y(x)/x \to 0$ *as* $x \to \infty$ *tends monotonically to zero as* $x \to \infty$.

Furthermore, if (E_{ii}) $[(E_{iii})]$ has two linearly independent solutions y_1 and y_2 such that

$$\lim_{x \to \infty} y_1(x)/x = \lim_{x \to \infty} y_2(x)/x = 0,$$

then (E_{ii}) $[(E_{iii})]$ is oscillatory. If either y_1 or y_2 is oscillatory, there is nothing to prove. Otherwise, consider the solution defined by $u(x) = y_2(a) y_1(x) - y_1(a) y_2(x)$. Since $\lim_{x \to \infty} u(x)/x = 0$ and $u(a) = 0$, the solution u is oscillatory by Corollary 3.

Every nonoscillatory solution $y = y(x)$ of (E_i) or (E_{ii}) tends to zero as $x \to \infty$, provided

$$\int_a^\infty x^{n-2}p(x)\,dx = \infty \tag{7}$$

or

$$p(x) \geqq (\bar{\lambda} + \varepsilon)/x^n, \qquad \varepsilon > 0, \tag{8}$$

where $\bar{\lambda}$ is defined to be the maximum of the local maxima of the function $f(x) \equiv -x(x-1)(x-2)\cdots(x-n+1)$ [1], [4]. Using this fact and Corollary 1, we can easily conclude that every solution of (E_i) is oscillatory if (7) or (8) is satisfied [1], [4]. Similarly, we can show that (E_{ii}) has a fundamental system consisting of n oscillatory solutions, provided either (7) or (8) is satisfied. For example, let y_i be the solution defined by the initial conditions $y_i^{(j-1)}(a) = \delta_{ij}$, $i, j = 1, 2, \ldots, n$. Then $\{y_1, y_2, \ldots, y_n\}$ is such a system (Corollary 3). Since equation (E_{ii}) is known to have a nonoscillatory solution [2], it also has a fundamental system consisting of one nonoscillatory solution and $n-1$ oscillatory solutions if (7) or (8) is fulfilled. Kondrat'ev obtained similar results for (E_{iii}) and (E_{iv}) [4].

References

1. G. V. Anan'eva and V. I. Balaganskiĭ, *Oscillation of the solutions of certain differential equations of high order*, Uspehi Mat. Nauk **14** (1959), no. 1 (85), 135–140. (Russian) MR **21** #1428.

2. P. Hartman and A. Wintner, *Linear differential and difference equations with monotone solutions*, Amer. J. Math. **75** (1953), 731–743. MR **15**, 221.

3. A. Kneser, *Untersuchen über die reellen Nullstellen der Integrale linearer Differentialgleichungen*, Math. Ann. **42** (1893), 409–435.

4. V. A. Kondrat'ev, *Oscillatory properties of solutions of the equation* $y^{(n)} + p(x)y = 0$, Trudy Moskov. Mat. Obšč. **10** (1961), 419–436. (Russian) MR **25** #5239.

5. W. Leighton, *The detection of the oscillation of solutions of a second order linear differential equation*, Duke Math. J. **17** (1950), 57–61. MR **11**, 248; 871.

6. Z. Nehari, *Non-oscillation criteria for n-th order linear differential equations*, Duke Math. J. **32** (1965), 607–615. MR **32** #4338.

7. ———, *Disconjugate linear differential operators*, Trans. Amer. Math. Soc. **129** (1967), 500–516. MR **36** #2860.

8. ———, *Disconjugacy criteria for linear differential equations*, J. Differential Equations **4** (1968), 604–611. MR **38** #1329.

9. T. T. Read, *Growth and decay of solutions of* $y^{(2n)} - py = 0$, Proc. Amer. Math. Soc. **43** (1974), 127–132. MR **49** #726.

10. W. Simons, *Monotonicity in some nonoscillation criteria for differential equations*, J. Differential Equations **13** (1973), 124–126. MR **48** #11671.

11. I. M. Sobol', *On the asymptotic behavior of the solutions of linear differential equations*, Dokl. Akad. Nauk SSSR (N.S.) **61** (1948), 219–222. (Russian) MR **10**, 40.

12. C. A. Swanson, *Comparison and oscillation theory of linear differential equations*, Academic Press, New York and London, 1968.

13. A. Wintner, *A criterion of oscillatory stability*, Quart. Appl. Math. **7** (1949), 115–117. MR **10**, 456.

Department of Applied Mathematics and Statistics, State University of New York, Stony Brook, New York 11794