

MONOTONE AND OSCILLATORY SOLUTIONS OF $y^{(n)} + py = 0$

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ABSTRACT. Monotone and oscillatory behaviors of the solutions with the property that $y(x)/x^2 \rightarrow 0$ as $x \rightarrow \infty$ or $y(x)/x \rightarrow 0$ as $x \rightarrow \infty$ are discussed. For example, it is shown that every nonoscillatory solution y , such that $y(x)/x \rightarrow 0$ as $x \rightarrow \infty$, monotonically tends to zero as $x \rightarrow \infty$, provided n is odd, $p \geq 0$, and $\int_a^\infty x^{n-1}p(x) dx = \infty$.

The differential equation to be considered is

$$(E) \quad y^{(n)} + py = 0,$$

where p is continuous and of one sign on an interval $[a, \infty)$. Since the behaviors of solutions of (E) depend on the parity of n and the sign of p , it is natural to classify (E) into the following four cases:

- (i) n even, $p \geq 0$,
- (ii) n odd, $p \geq 0$,
- (iii) n even, $p \leq 0$,
- (iv) n odd, $p \leq 0$.

In the sequel, (E_i) , for example, shall denote the equation (E) satisfying condition (i). Likewise, (E_{ii}) , (E_{iii}) , and (E_{iv}) denote equation (E) satisfying (ii), (iii), and (iv), respectively.

A nontrivial solution of (E) is said to be *oscillatory* on $[a, \infty)$ if it has an infinity of zeros on $[a, \infty)$; otherwise, it is said to be *nonoscillatory* on $[a, \infty)$. If every solution of (E) is nonoscillatory on $[a, \infty)$, (E) is said to be *nonoscillatory* on $[a, \infty)$. On the other hand, if (E) has an oscillatory solution, (E) is said to be *oscillatory*.

Oscillatory and nonoscillatory behaviors of (E) have been extensively studied by a number of authors [1]–[13]. In particular, Anan'eva and Balagan-ski [1] proved the following statements: If $p > 0$ and $\int_a^\infty x^{n-2}p(x) dx = \infty$, then every solution of (E) is oscillatory if n is even, while every nonoscillatory

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solution of (E) with odd n satisfies

$$\lim_{x \rightarrow \infty} y^{(n-1)}(x) = \dots = \lim_{x \rightarrow \infty} y'(x) = \lim_{x \rightarrow \infty} y(x) = 0,$$

where the signs of $y, y', \dots, y^{(n)}$ are preserved for sufficiently large x and successively alternating. Similar results were obtained by Kondrat'ev [4] under different conditions imposed on the coefficient p . Central to the proofs of these results is that any solution y with $y(x) \rightarrow 0$ as $x \rightarrow \infty$ is oscillatory, provided equation (E) meets certain requirements.

It is our aim to improve and extend some aspects of the above results. For example, it will be shown that every nontrivial solution y of (E_i) or (E_{iv}) such that $y(x)/x^2 \rightarrow 0$ as $x \rightarrow \infty$, either is oscillatory or satisfies a sequence of inequalities, which has interesting consequences. We shall also prove that every nonoscillatory solution y of (E_{ii}) [(E_{iii})] such that $y(x)/x \rightarrow 0$ as $x \rightarrow \infty$ tends to zero as $x \rightarrow \infty$ if $\int_a^\infty x^{n-1} p(x) dx = \infty$ [$-\infty$].

THEOREM 1. Assume that $p \neq 0$ on $[a_1, \infty)$ for every $a_1 \geq a$. If y is a nontrivial solution of (E_i) or (E_{iv}) such that $y \geq 0$ and $y(x)/x^2 \rightarrow 0$ as $x \rightarrow \infty$, then

$$(1) \quad \begin{aligned} y(x) &\geq 0, \quad y'(x) > 0, \quad y''(x) < 0, \\ y'''(x) &> 0, \dots, (-1)^n y^{(n-1)}(x) > 0, \quad x \in [a, \infty), \end{aligned}$$

and $y^{(k)}(x) \rightarrow 0$ monotonically as $x \rightarrow \infty$, $k = 2, 3, \dots, n-1$.

PROOF. Put $y_1 = y, y_2 = y', \dots, y_n = y^{(n-1)}$ and let b be an arbitrary point of $[a, \infty)$. Then y satisfies the system

$$\begin{aligned} y_1(x) &= y_1(b) + \int_b^x y_2(t) dt, \\ &\dots \\ y_{n-1}(x) &= y_{n-1}(b) + \int_b^x y_n(t) dt, \\ y_n(x) &= y_n(b) - \int_b^x p(t) y_1(t) dt. \end{aligned}$$

Suppose $y = y_1$ is a solution of (E_i) . Then $\int_b^x p(t) y_1(t) dt$ is a nondecreasing, nonnegative function of x and clearly is positive on an interval $[c, \infty)$ for some $c > b$. We claim that $y_n(b) > 0$. To prove this, assume the contrary: $y_n(b) \leq 0$. Then, $y_n(x)$ is nonpositive, nonincreasing on $[b, \infty)$ and

$$y_n(c) = y_n(b) - \int_b^c p(t) y_1(t) dt < 0,$$

that is, $y_n(x) \leq y_n(c) < 0$, $x \in [c, \infty)$. Consequently, $y_{n-1}(x) \rightarrow -\infty$ as $x \rightarrow \infty$, irrespective of $y_{n-1}(b)$. This in turn implies $y_{n-2}(x) \rightarrow -\infty$ as $x \rightarrow \infty$, and successively $y_k(x) \rightarrow -\infty$ as $x \rightarrow \infty$, regardless of the value $y_k(b)$, $k = 1, 2, \dots, n-1$. In particular, $y_1(x) = y(x) \rightarrow -\infty$ as $x \rightarrow \infty$, contrary to the

hypothesis that $y \geq 0$ on $[a, \infty)$. This contradiction proves $y_n(b) > 0$. Since b is arbitrary, we conclude that $y_n(x) > 0$, $x \in [a, \infty)$. It is now easy to see that $y_n(x) \rightarrow 0$ as $x \rightarrow \infty$ for $n > 2$. If this were not the case, there would exist a constant $K > 0$ such that $y_n(x) > K$, $x \in [c_1, \infty)$, for some $c_1 \geq a$. However, this implies that $y(x) > K_1 x^{n-1}$ on $[c_2, \infty)$ for some constants $K_1 > 0$ and $c_2 > c_1$, contradicting the asymptotic behavior $y(x)/x^2 \rightarrow 0$ as $x \rightarrow \infty$. Next, we shall prove that $y_{n-1}(x) < 0$ if $n > 2$. Evidently, $y_{n-1}(x)$ is a monotonically increasing function. If $y_{n-1}(b) \geq 0$, then $y_{n-1}(x) \geq 0$ on $[b, \infty)$, and there would exist constants $C > 0$ and $d > b$ such that $y_{n-1}(x) > C$, $x \in [d, \infty)$. However, for $n > 2$, this again leads to the contradiction that $y(x) > Cx^{n-2}$, $x \in [d_1, \infty)$, for some $d_1 > d$. Thus, $y_{n-1}(b) < 0$, and $y_{n-1}(x) < 0$ since b is arbitrary. Moreover, we must have $y_{n-1}(x) \rightarrow 0$ as $x \rightarrow \infty$, for otherwise we would again be led to the contradiction that $y(x) \rightarrow -\infty$ as $x \rightarrow \infty$. In this way, we can successively establish the inequalities

$$y_n(x) > 0, \quad y_{n-1}(x) < 0, \quad \dots, \quad y_4(x) > 0, \quad y_3(x) < 0, \quad x \in [a, \infty),$$

with the property that $y_k(x) \rightarrow 0$ as $x \rightarrow \infty$, $k = 3, 4, \dots, n$. Continuing this process, we deduce $y_2(x) > 0$ and $y_1(x) \geq 0$, $x \in [a, \infty)$. This proves the theorem for (E_i) . The proof for (E_{iv}) is similar; in this case, we first prove that $y_n(x) < 0$ and $y_n(x) \rightarrow 0$ as $x \rightarrow \infty$, and continue as in the case of (E_i) .

In a somewhat similar fashion, we can prove

THEOREM 2. Assume that $p \neq 0$ on $[a_1, \infty)$ for every $a_1 \geq a$. If y is a nontrivial solution of (E_{ii}) or (E_{iii}) such that $y \geq 0$ on $[a, \infty)$ and $y(x)/x \rightarrow 0$ as $x \rightarrow \infty$, then

$$(2) \quad y(x) > 0, \quad y'(x) < 0, \quad y''(x) > 0, \quad \dots, \quad (-1)^n y^{(n-1)}(x) < 0, \\ x \in [a, \infty),$$

and $y^{(k)}(x) \rightarrow 0$ monotonically as $x \rightarrow \infty$, $k = 1, 2, \dots, n-1$.

In order to characterize the behaviors of solutions, we may reformulate Theorem 1 as follows:

COROLLARY 1. Suppose $p \neq 0$ on $[a_1, \infty)$ for every $a_1 \geq a$. Let y be a nontrivial solution of (E_i) or (E_{iv}) such that $y(x)/x^2 \rightarrow 0$ as $x \rightarrow \infty$. Then either

- (a) y is oscillatory on $[a, \infty)$, or else
- (b) $y \geq 0$ [≤ 0] on $[b, \infty)$, for some $b \geq a$, and y [$-y$] satisfies the inequalities in (1) of Theorem 1. In particular, y [$-y$] increases [decreases] monotonically on $[b, \infty)$.

If y is a nontrivial solution of (E_i) or (E_{iv}) such that $y(x) \rightarrow 0$ as $x \rightarrow \infty$, it cannot satisfy the inequalities in (1) of Theorem 1. Thus, we conclude by Corollary 1 that y is oscillatory.

THEOREM 3. If $\int_a^\infty x^{n-1} p(x) dx = \infty$ [$-\infty$], every nonoscillatory solution of (E_i) [(E_{iv})] is unbounded on $[a, \infty)$.

PROOF. Assume the contrary. Suppose there exists a nontrivial solution y which is bounded and positive on $[b, \infty)$, for some $b \geq a$. Since y increases monotonically by Theorem 1, there exist two positive constants M_1 and M_2 such that $M_1 \leq y(x) \leq M_2$, $x \in [b, \infty)$. Multiplying (E) by x^{n-1} and integrating n times by parts, we get

$$(3) \quad \begin{aligned} &x^{n-1}y^{(n-1)} - (n-1)x^{n-2}y^{(n-2)} + (n-1)(n-2)x^{n-3}y^{(n-3)} \\ &\quad - \dots + (-1)^{n-1}(n-1)!y = - \int_b^x t^{n-1}p(t)y(t) dt + C, \end{aligned}$$

where C is a constant.

If y is a solution of (E_i) , then n is even, $p \geq 0$, and we get from (3),

$$(4) \quad \begin{aligned} &x^{n-1}y^{(n-1)} - (n-1)x^{n-2}y^{(n-2)} + \dots \\ &\quad + [(n-1)(n-2) \dots 2]xy' - (n-1)!M_2 \\ &\quad < -M_1 \int_b^x t^{n-1}p(t) dt + C, \quad x \in [b, \infty). \end{aligned}$$

On the other hand, since $y^{(n-1)} > 0$, $y^{(n-2)} < 0$, \dots , $y' > 0$ by Theorem 1, the left-hand side of (4) cannot tend to $-\infty$ as $x \rightarrow \infty$, while the right-hand side tends to $-\infty$ as $x \rightarrow \infty$. Therefore, inequality (4) cannot hold throughout $[b, \infty)$. This incompatibility proves that the solution y must be unbounded on $[a, \infty)$.

If y is a solution of (E_{iv}) , then n is odd, $p \leq 0$, and we get from (3),

$$(5) \quad \begin{aligned} &x^{n-1}y^{(n-1)} - (n-1)x^{n-2}y^{(n-2)} + \dots \\ &\quad - [(n-1)(n-2) \dots 2]xy' + (n-1)!M_2 \\ &\quad > -M_1 \int_b^x t^{n-1}p(t) dt + C, \quad x \in [b, \infty). \end{aligned}$$

In this case, $y^{(n-1)} < 0$, $y^{(n-2)} > 0$, \dots , $y' > 0$, due to Theorem 1. For this reason, the left-hand side of (5) cannot approach ∞ as $x \rightarrow \infty$, while the right-hand side approaches ∞ as $x \rightarrow \infty$. Hence, inequality (5) cannot hold throughout $[b, \infty)$, and our assumption that y is bounded must be false.

Theorem 3 may be restated as follows: If (E_i) [(E_{iv})] has a bounded nonoscillatory solution on $[a, \infty)$, then $\int_a^\infty x^{n-1}p(x) dx < \infty$ [$> -\infty$]. However, the above inequality is known to guarantee the nonoscillation of (E_i) [(E_{iv})] on $[a, \infty)$ [4], [11]. Thus, we have the following result.

COROLLARY 2. *If (E_i) [(E_{iv})] has a bounded nonoscillatory solution defined on $[a, \infty)$, then (E_i) [(E_{iv})] is nonoscillatory on $[a, \infty)$.*

According to a result of Kondrat'ev [4], any solution y of $y^{(n)} + py = 0$, $p \in C[a, \infty)$, $p \geq 0$, such that $y(x) \rightarrow 0$ as $x \rightarrow \infty$ and $y(b) = 0$, for some $b \in [a, \infty)$, is oscillatory on $[a, \infty)$. In this connection, we have the following corollary of Theorem 2.

COROLLARY 3. *Suppose $p \not\equiv 0$ on $[a_1, \infty)$ for every $a_1 \geq a$. Let y be a nontrivial solution of (E_{ii}) or (E_{iii}) such that $y(x)/x \rightarrow 0$ as $x \rightarrow \infty$. If y violates any of the inequalities in (2) of Theorem 2, then it is oscillatory on $[a, \infty)$. In particular, if $y^{(k)}(b) = 0$ for some k , $0 \leq k \leq n - 1$, and some point $b \in [a, \infty)$, then y is oscillatory on $[a, \infty)$.*

It is well known that equation (E_{ii}) and (E_{iii}) have a nontrivial solution y satisfying the inequalities

$$(6) \quad (-1)^j y^{(j)}(x) \geq 0, \quad j = 0, 1, \dots, n - 1, \quad x \in [a, \infty),$$

which is due to Hartman and Wintner [2]. In a recent paper [9, Theorem 4], Read showed that any solution $z = z(x)$ of (E_{iii}) satisfying (6), tends to zero as $x \rightarrow \infty$ if and only if $\int_a^\infty t^{n-1} p(t) dt = -\infty$. We shall establish a similar result for (E_{ii}) .

LEMMA. *Suppose y is a nontrivial solution of (E_{ii}) which satisfies (6). Then $y(x) \rightarrow 0$ as $x \rightarrow \infty$ if $\int_a^\infty x^{n-1} p(x) dx = \infty$.*

PROOF. Suppose $y(x)$ does not approach 0 as $x \rightarrow \infty$. Then there exists a positive constant M such that $y(x) > M$, $x \in [a, \infty)$. Using (3) with b replaced by a , and recalling that n is odd and $p \geq 0$, we deduce

$$\begin{aligned} & x^{n-1} y^{(n-1)} - (n-1)x^{n-2} y^{(n-2)} + \dots \\ & \quad - [(n-1)(n-2) \cdots 2] x y' + (n-1)! M \\ & < -M \int_a^x t^{n-1} p(t) dt + C_1, \quad x \in [a, \infty), \end{aligned}$$

where C_1 is a constant. Since the left-hand side of this inequality is positive on $[a, \infty)$ because of (6), $\int_a^x t^{n-1} p(t) dt$ cannot tend to ∞ as $x \rightarrow \infty$. This proves the Lemma.

In view of Theorem 2, Theorem 4 of Read [9], and the Lemma, we have

THEOREM 4. *If $\int_a^\infty x^{n-1} p(x) dx = \infty [-\infty]$, every nonoscillatory solution y of (E_{ii}) [(E_{iii})] such that $y(x)/x \rightarrow 0$ as $x \rightarrow \infty$ tends monotonically to zero as $x \rightarrow \infty$.*

Furthermore, if (E_{ii}) [(E_{iii})] has two linearly independent solutions y_1 and y_2 such that

$$\lim_{x \rightarrow \infty} y_1(x)/x = \lim_{x \rightarrow \infty} y_2(x)/x = 0,$$

then (E_{ii}) [(E_{iii})] is oscillatory. If either y_1 or y_2 is oscillatory, there is nothing to prove. Otherwise, consider the solution defined by $u(x) = y_2(a)y_1(x) - y_1(a)y_2(x)$. Since $\lim_{x \rightarrow \infty} u(x)/x = 0$ and $u(a) = 0$, the solution u is oscillatory by Corollary 3.

Every nonoscillatory solution $y = y(x)$ of (E_i) or (E_{ii}) tends to zero as $x \rightarrow \infty$, provided

$$(7) \quad \int_a^\infty x^{n-2} p(x) dx = \infty$$

or

$$(8) \quad p(x) \geq (\bar{\lambda} + \varepsilon)/x^n, \quad \varepsilon > 0,$$

where $\bar{\lambda}$ is defined to be the maximum of the local maxima of the function $f(x) \equiv -x(x-1)(x-2)\cdots(x-n+1)$ [1], [4]. Using this fact and Corollary 1, we can easily conclude that every solution of (E_i) is oscillatory if (7) or (8) is satisfied [1], [4]. Similarly, we can show that (E_{ii}) has a fundamental system consisting of n oscillatory solutions, provided either (7) or (8) is satisfied. For example, let y_i be the solution defined by the initial conditions $y_i^{(j-1)}(a) = \delta_{ij}$, $i, j = 1, 2, \dots, n$. Then $\{y_1, y_2, \dots, y_n\}$ is such a system (Corollary 3). Since equation (E_{ii}) is known to have a nonoscillatory solution [2], it also has a fundamental system consisting of one nonoscillatory solution and $n-1$ oscillatory solutions if (7) or (8) is fulfilled. Kondrat'ev obtained similar results for (E_{iii}) and (E_{iv}) [4].

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