

ON THE EXTENSIONS TO THE BIDUAL OF A MAXIMAL MONOTONE OPERATOR

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ABSTRACT. An example is given which shows that a maximal monotone operator from a Banach space X to its dual X^* may have several extensions into a maximal monotone operator from X^{**} to X^* .

Introduction. Let X be a real Banach space with dual X^* and let $T: X \rightarrow 2^{X^*}$ be a maximal monotone operator. Identifying as usual X to a subspace of X^{**} , we look at T as a monotone operator from X^{**} to 2^{X^*} ; by Zorn's lemma, this operator can be extended into a maximal monotone operator from X^{**} to 2^{X^*} . We are interested here in the question whether this extension is *unique*.

There are a number of cases where it is so.

Denote by $\bar{T}: X^{**} \rightarrow 2^{X^*}$ the (monotone) operator whose graph is the closure of the graph of T with respect to the weakest topology on $X^{**} \times X^*$ which is stronger than $\sigma(X^{**}, X^*) \times \sigma(X^*, X^{**})$ and such that $(x^{**}, x^*) \rightarrow \langle x^{**}, x^* \rangle$ is upper semicontinuous. Since any maximal monotone extension of T to X^{**} contains \bar{T} , we see that if \bar{T} is maximal monotone, then T has a unique maximal monotone extension to X^{**} . An operator T such that \bar{T} is maximal monotone is called *of dense type* (a terminology slightly different from that of [2]). This kind of condition arises in the study of monotone operators in nonreflexive Banach spaces (cf. [2], [1], [7]). It is known, for instance, that the subdifferential of a convex function or the monotone operator associated with a saddle function are of dense type (cf. [6], [2], [5], [4]).

On the other hand, there are maximal monotone operators which are not of dense type but which have a unique maximal monotone extension to the bidual (cf. the example in [3]: the uniqueness assertion is contained in Proposition 1 of [3] and the fact that the operator considered there is not of dense type follows easily from relation (1) of [3]).

It is our purpose in this note to construct a maximal monotone operator which admits several (actually an infinity) maximal monotone extensions to the bidual. Our construction is based on a refinement of the method used in [3].

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Example. Let $A: l^1 \rightarrow l^\infty$ be the bounded linear operator defined by

$$(1) \quad (Ax)(n) = \sum_{m=1}^{\infty} x(m)K(m, n)$$

for $x = (x(1), x(2), \dots) \in l^1$, where the infinite matrix K is constructed in the following way: take a bounded sequence (r_1, r_2, \dots) of real numbers in which each number $-n^{-2}$ ($n = 1, 2, \dots$) appears infinitely many times and write

$$(2) \quad K = \begin{bmatrix} 0 & r_1 & r_2 & r_3 & \cdots \\ -r_1 & 0 & r_2 & r_3 & \cdots \\ -r_2 & -r_2 & 0 & r_3 & \cdots \\ -r_3 & -r_3 & -r_3 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

The corresponding operator A is then antisymmetric (i.e. $A \subset -A^*$) and thus (maximal) monotone.

PROPOSITION. *There are infinitely many maximal monotone operators $B: (l^\infty)^* \rightarrow 2^{l^\infty}$ which extend A .*

Let us say that two points (x^{**}, x^*) and (y^{**}, y^*) in $X^{**} \times X^*$ are monotonely related if $\langle x^{**} - y^{**}, x^* - y^* \rangle \geq 0$ and that a point (x^{**}, x^*) is monotonely related to a subset of $X^{**} \times X^*$ if it is monotonely related to each point of this subset. Clearly, by Zorn's lemma, the proposition will be proved if we exhibit an infinite number of points in $(l^\infty)^* \times l^\infty$ such that each of them is monotonely related to the graph of A but any two of them are not monotonely related.

LEMMA 1 (cf. [3]). *Let $A: X \rightarrow X^*$ be a bounded linear antisymmetric operator. Then (x^{**}, x^*) is monotonely related to the graph of A if and only if $x^* = -A^*x^{**}$ and $\langle x^{**}, x^* \rangle \geq 0$.*

PROOF. Let (x^{**}, x^*) verify $\langle x^{**} - y, x^* - Ay \rangle \geq 0$ for all $y \in X$. Then

$$\langle x^{**}, x^* \rangle \geq \langle y, x^* \rangle + \langle x^{**}, Ay \rangle$$

for all $y \in X$, which implies $\langle x^{**}, x^* \rangle \geq 0$ and $x^* = -A^*x^{**}$. The converse implication is proved by direct calculation. Q.E.D.

Let $\beta\mathbb{N}$ denote the Stone-Ćech compactification of \mathbb{N} ; then l^∞ can be identified to the space $C(\beta\mathbb{N})$ of the continuous real-valued functions on $\beta\mathbb{N}$ and $(l^\infty)^*$ to the space $\mathfrak{M}(\beta\mathbb{N})$ of the Radon measures on $\beta\mathbb{N}$. Given a bounded infinite matrix K , we consider for $m \in \mathbb{N}$ the function $K(m, \cdot)$ on \mathbb{N} and extend it continuously on $\beta\mathbb{N}$; let $K(m, a)$ denote the value of this extension at $a \in \beta\mathbb{N}$. Then we consider for $a \in \beta\mathbb{N}$ the function $K(\cdot, a)$ on \mathbb{N} and extend it continuously on $\beta\mathbb{N}$; let $K(b, a)$ denote the value of this

extension at $b \in \beta\mathbb{N}$. If K is antisymmetric, then the extended matrix $K(b, a)$ verifies $K(m, a) = -K(a, m)$ for $a \in \beta\mathbb{N}$ and $m \in \mathbb{N}$, but is generally not antisymmetric on $\beta\mathbb{N} \times \beta\mathbb{N}$, as is illustrated by the following simple example (cf. [3]):

$$(3) \quad K(m, n) = 0 \text{ if } m = n, \quad -1 \text{ if } n > m, \quad +1 \text{ if } n < m;$$

this example also gives some feeling for formula (5) below.

We will assume below that

$$(4) \quad \text{for any } a \in \beta\mathbb{N} \setminus \mathbb{N}, K(m, a) \text{ converges as } m \rightarrow \infty;$$

this means that for any $a \in \beta\mathbb{N} \setminus \mathbb{N}$, $K(b, a)$ as a function of b is constant on $\beta\mathbb{N} \setminus \mathbb{N}$. This condition is satisfied by the matrices (2) and (3).

LEMMA 2. Let $A: l^1 \rightarrow l^\infty$ be a bounded linear antisymmetric operator with an associated matrix K satisfying (4). Then

$$(5) \quad \langle \mu, -A^* \mu \rangle = -\mu(\beta\mathbb{N} \setminus \mathbb{N}) \cdot \int_{\beta\mathbb{N} \setminus \mathbb{N}} K(b, a) d\mu(a)$$

for all $\mu \in M(\beta\mathbb{N})$, where b in the right-hand side is arbitrary in $\beta\mathbb{N} \setminus \mathbb{N}$.

PROOF. First we deduce from (1) that

$$(6) \quad (Ax)(a) = \sum_{m=1}^{\infty} x(m)K(m, a)$$

for $x \in l^1$ and $a \in \beta\mathbb{N}$. Indeed, if $n_i \in \mathbb{N}$ is a generalized sequence converging to a , then $K(\cdot, n_i)$ remains bounded in l^∞ and converges componentwise to $K(\cdot, a)$; consequently $K(\cdot, n_i)$ converges to $K(\cdot, a)$ in $l^\infty, \sigma(l^\infty, l^1)$, and (6) follows from (1). Now we have

$$(7) \quad (A^* \mu)(m) = \int_{\beta\mathbb{N}} K(m, a) d\mu(a)$$

for $\mu \in \mathfrak{M}(\beta\mathbb{N})$ and $m \in \mathbb{N}$. Indeed, for any $y \in l^1$,

$$\langle y, A^* \mu \rangle = \langle \mu, Ay \rangle = \int_{\beta\mathbb{N}} \left[\int_{\mathbb{N}} y(m)K(m, a) d\nu(m) \right] d\mu(a)$$

where ν denotes the counting measure on \mathbb{N} ; equality (7) then follows from Fubini's theorem. Under assumption (4), we have

$$(8) \quad (A^* \mu)(b) = \int_{\beta\mathbb{N}} K(b, a) d\mu(a)$$

for $\mu \in \mathfrak{M}(\beta\mathbb{N})$ and $b \in \beta\mathbb{N}$. Indeed (7) gives

$$(A^* \mu)(m) = \int_{\mathbb{N}} K(m, a) d\mu(a) + \int_{\beta\mathbb{N} \setminus \mathbb{N}} K(m, a) d\mu(a);$$

by an argument similar to the preceding one involving $\sigma(l^\infty, l^1)$, we can pass

to the limit in the first integral, and by Lebesgue's theorem, using assumption (4), we can pass to the limit in the second integral. Finally, for $\mu \in \mathfrak{N}(\beta\mathbb{N})$, we have

$$\langle \mu, -A^* \mu \rangle = - \int_{\beta\mathbb{N}} \left[\int_{\beta\mathbb{N}} K(b, a) d\mu(a) \right] d\mu(b);$$

writing each integral in the right-hand side as the sum of an integral over \mathbb{N} and an integral over $\beta\mathbb{N} \setminus \mathbb{N}$, using the antisymmetry of K and Fubini's theorem, we obtain (5). Q.E.D.

PROOF OF THE PROPOSITION. From the choice of K (cf. (2)), we can find a sequence a_n in $\beta\mathbb{N} \setminus \mathbb{N}$ such that $K(b, a_n) = -n^{-2}$ for $b \in \beta\mathbb{N} \setminus \mathbb{N}$. (Recall that when $a \in \beta\mathbb{N} \setminus \mathbb{N}$, $K(b, a)$ as a function of b is constant on $\beta\mathbb{N} \setminus \mathbb{N}$.) Let $\mu_n = n\delta_{a_n}$ where δ_{a_n} denotes the Dirac measure at the point a_n . Using Lemma 1, we see that each $(\mu_n, -A^* \mu_n)$ is monotonely related to the graph of A since, by Lemma 2,

$$\langle \mu_n, -A^* \mu_n \rangle = -n \cdot (-n^{-2}) \cdot n \geq 0.$$

But if $n \neq m$, then $(\mu_n, -A^* \mu_n)$ and $(\mu_m, -A^* \mu_m)$ are not monotonely related since, by Lemma 2,

$$\begin{aligned} \langle \mu_n - \mu_m, -A^* \mu_n + A^* \mu_m \rangle &= \langle \mu_n - \mu_m, -A^* (\mu_n - \mu_m) \rangle \\ &= -(n - m) \cdot (-n^{-2} \cdot n + m^{-2} \cdot m) < 0, \end{aligned}$$

which concludes the proof. Q.E.D.

REMARK. In the situation of the above Proposition, there are infinitely many linear maximal monotone operators $B: D(B) \subset (l^\infty)^* \rightarrow l^\infty$ which extend A . This follows easily from our construction and from the following two simple facts: (a) in a dual pair (E, F) , let $T: D(T) \subset E \rightarrow F$ be a linear monotone operator and let $(e, f) \in E \times F$ be monotonely related to the graph of T , with $e \notin D(T)$; then the linear extension of T to $\text{vect}[D(T), e]$:

$$\tilde{T}(x + \lambda e) = T(x) + \lambda f \quad \text{for } x \in D(T) \text{ and } \lambda \in \mathbf{R},$$

is still monotone; (b) let $T: D(T) \subset E \rightarrow F$ be a linear monotone operator with a $\sigma(E, F)$ dense domain; if T is maximal among all linear monotone operators, then T is maximal monotone.

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