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ON THE EXTENSIONS TO THE BIDUAL OF A MAXIMAL MONOTONE OPERATOR

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ABSTRACT. An example is given which shows that a maximal monotone operator from a Banach space X to its dual X^* may have several extensions into a maximal monotone operator from X^{**} to X^*.

Introduction. Let X be a real Banach space with dual X^* and let $T: X \to 2^{X^*}$ be a maximal monotone operator. Identifying as usual X to a subspace of X^{**}, we look at T as a monotone operator from X^{**} to 2^{X^*}; by Zorn's lemma, this operator can be extended into a maximal monotone operator from X^{**} to 2^{X^*}. We are interested here in the question whether this extension is *unique*.

There are a number of cases where it is so.

Denote by $\overline{T}: X^{**} \to 2^{X^*}$ the (monotone) operator whose graph is the closure of the graph of T with respect to the weakest topology on $X^{**} \times X^*$ which is stronger than $\sigma(X^{**}, X^*) \times \sigma(X^*, X^{**})$ and such that $(x^{**}, x^*) \to \langle x^{**}, x^* \rangle$ is upper semicontinuous. Since any maximal monotone extension of T to X^{**} contains $\overline{T}$, we see that if $\overline{T}$ is maximal monotone, then T has a unique maximal monotone extension to X^{**}. An operator T such that $\overline{T}$ is maximal monotone is called *of dense type* (a terminology slightly different from that of [2]). This kind of condition arises in the study of monotone operators in nonreflexive Banach spaces (cf. [2], [1], [7]). It is known, for instance, that the subdifferential of a convex function or the monotone operator associated with a saddle function are of dense type (cf. [6], [2], [5], [4]).

On the other hand, there are maximal monotone operators which are not of dense type but which have a unique maximal monotone extension to the bidual (cf. the example in [3]: the uniqueness assertion is contained in Proposition 1 of [3] and the fact that the operator considered there is not of dense type follows easily from relation (1) of [3]).

It is our purpose in this note to construct a maximal monotone operator which admits several (actually an infinity) maximal monotone extensions to the bidual. Our construction is based on a refinement of the method used in [3].

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Example. Let $A\colon l^1 \to l^\infty$ be the bounded linear operator defined by

$$(1) \qquad (Ax)(n) = \sum_{m=1}^{\infty} x(m)K(m,n)$$

for $x = (x(1), x(2), \dots) \in l^1$, where the infinite matrix K is constructed in the following way: take a bounded sequence $(r_1, r_2, \dots)$ of real numbers in which each number $-n^{-2}$ ($n = 1, 2, \dots$) appears infinitely many times and write

$$(2) \qquad K = \begin{bmatrix} 0 & r_1 & r_2 & r_3 & \cdots \\ -r_1 & 0 & r_2 & r_3 & \cdots \\ -r_2 & -r_2 & 0 & r_3 & \cdots \\ -r_3 & -r_3 & -r_3 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

The corresponding operator A is then antisymmetric (i.e. $A \subset -A^*$) and thus (maximal) monotone.

PROPOSITION. *There are infinitely many maximal monotone operators $B\colon (l^\infty)^* \to 2^{l^\infty}$ which extend A.*

Let us say that two points (x^{**}, x^*) and (y^{**}, y^*) in $X^{**} \times X^*$ are monotonely related if $\langle x^{**} - y^{**}, x^* - y^* \rangle \geqslant 0$ and that a point (x^{**}, x^*) is monotonely related to a subset of $X^{**} \times X^*$ if it is monotonely related to each point of this subset. Clearly, by Zorn's lemma, the proposition will be proved if we exhibit an infinite number of points in $(l^\infty)^* \times l^\infty$ such that each of them is monotonely related to the graph of A but any two of them are not monotonely related.

LEMMA 1 (cf. [3]). *Let $A\colon X \to X^*$ be a bounded linear antisymmetric operator. Then (x^{**}, x^*) is monotonely related to the graph of A if and only if $x^* = -A^* x^{**}$ and $\langle x^{**}, x^* \rangle \geqslant 0$.*

PROOF. Let (x^{**}, x^*) verify $\langle x^{**} - y, x^* - Ay \rangle \geqslant 0$ for all $y \in X$. Then

$$\langle x^{**}, x^* \rangle \geqslant \langle y, x^* \rangle + \langle x^{**}, Ay \rangle$$

for all $y \in X$, which implies $\langle x^{**}, x^* \rangle \geqslant 0$ and $x^* = -A^* x^{**}$. The converse implication is proved by direct calculation. Q.E.D.

Let $\beta\mathbf{N}$ denote the Stone-Čech compactification of $\mathbf{N}$; then l^∞ can be identified to the space $C(\beta\mathbf{N})$ of the continuous real-valued functions on $\beta\mathbf{N}$ and $(l^\infty)^*$ to the space $\mathfrak{M}(\beta\mathbf{N})$ of the Radon measures on $\beta\mathbf{N}$. Given a bounded infinite matrix K, we consider for $m \in \mathbf{N}$ the function $K(m, \cdot)$ on $\mathbf{N}$ and extend it continuously on $\beta\mathbf{N}$; let $K(m,a)$ denote the value of this extension at $a \in \beta\mathbf{N}$. Then we consider for $a \in \beta\mathbf{N}$ the function $K(\cdot, a)$ on $\mathbf{N}$ and extend it continuously on $\beta\mathbf{N}$; let $K(b,a)$ denote the value of this extension at $b \in \beta\mathbf{N}$. If K is antisymmetric, then the extended matrix $K(b,a)$ verifies $K(m,a) = -K(a,m)$ for $a \in \beta\mathbf{N}$ and $m \in \mathbf{N}$, but is generally not antisymmetric on $\beta\mathbf{N} \times \beta\mathbf{N}$, as is illustrated by the following simple example (cf. [3]):

$$K(m,n) = 0 \text{ if } m = n, \quad -1 \text{ if } n > m, \quad +1 \text{ if } n < m; \tag{3}$$

this example also gives some feeling for formula (5) below.

We will assume below that

$$\text{for any } a \in \beta\mathbf{N}\backslash\mathbf{N},\ K(m,a) \text{ converges as } m \to \infty; \tag{4}$$

this means that for any $a \in \beta\mathbf{N}\backslash\mathbf{N}$, $K(b,a)$ as a function of b is constant on $\beta\mathbf{N}\backslash\mathbf{N}$. This condition is satisfied by the matrices (2) and (3).

LEMMA 2. *Let $A\colon l^1 \to l^\infty$ be a bounded linear antisymmetric operator with an associated matrix K satisfying* (4). *Then*

$$\langle \mu, -A^* \mu \rangle = -\mu(\beta\mathbf{N}\backslash\mathbf{N}) \cdot \int_{\beta\mathbf{N}\backslash\mathbf{N}} K(b,a)\, d\mu(a) \tag{5}$$

for all $\mu \in M(\beta\mathbf{N})$, where b in the right-hand side is arbitrary in $\beta\mathbf{N}\backslash\mathbf{N}$.

PROOF. First we deduce from (1) that

$$(Ax)(a) = \sum_{m=1}^{\infty} x(m) K(m,a) \tag{6}$$

for $x \in l^1$ and $a \in \beta\mathbf{N}$. Indeed, if $n_i \in \mathbf{N}$ is a generalized sequence converging to a, then $K(\cdot, n_i)$ remains bounded in l^∞ and converges componentwise to $K(\cdot, a)$; consequently $K(\cdot, n_i)$ converges to $K(\cdot, a)$ in l^∞, $\sigma(l^\infty, l^1)$, and (6) follows from (1). Now we have

$$(A^* \mu)(m) = \int_{\beta\mathbf{N}} K(m,a)\, d\mu(a) \tag{7}$$

for $\mu \in \mathfrak{M}(\beta\mathbf{N})$ and $m \in \mathbf{N}$. Indeed, for any $y \in l^1$,

$$\langle y, A^* \mu \rangle = \langle \mu, Ay \rangle = \int_{\beta\mathbf{N}} \left[\int_{\mathbf{N}} y(m) K(m,a)\, d\nu(m) \right] d\mu(a)$$

where ν denotes the counting measure on $\mathbf{N}$; equality (7) then follows from Fubini's theorem. Under assumption (4), we have

$$(A^* \mu)(b) = \int_{\beta\mathbf{N}} K(b,a)\, d\mu(a) \tag{8}$$

for $\mu \in \mathfrak{M}(\beta\mathbf{N})$ and $b \in \beta\mathbf{N}$. Indeed (7) gives

$$(A^* \mu)(m) = \int_{\mathbf{N}} K(m,a)\, d\mu(a) + \int_{\beta\mathbf{N}\backslash\mathbf{N}} K(m,a)\, d\mu(a);$$

by an argument similar to the preceding one involving $\sigma(l^\infty, l^1)$, we can pass to the limit in the first integral, and by Lebesgue's theorem, using assumption (4), we can pass to the limit in the second integral. Finally, for $\mu \in \mathfrak{M}(\beta\mathbf{N})$, we have

$$\langle \mu, -A^* \mu \rangle = -\int_{\beta\mathbf{N}} \left[\int_{\beta\mathbf{N}} K(b,a)\, d\mu(a) \right] d\mu(b);$$

writing each integral in the right-hand side as the sum of an integral over $\mathbf{N}$ and an integral over $\beta\mathbf{N}\backslash\mathbf{N}$, using the antisymmetry of K and Fubini's theorem, we obtain (5). Q.E.D.

Proof of the Proposition. From the choice of K (cf. (2)), we can find a sequence a_n in $\beta\mathbf{N}\backslash\mathbf{N}$ such that $K(b, a_n) = -n^{-2}$ for $b \in \beta\mathbf{N}\backslash\mathbf{N}$. (Recall that when $a \in \beta\mathbf{N}\backslash\mathbf{N}$, $K(b,a)$ as a function of b is constant on $\beta\mathbf{N}\backslash\mathbf{N}$.) Let $\mu_n = n\delta_{a_n}$ where δ_{a_n} denotes the Dirac measure at the point a_n. Using Lemma 1, we see that each $(\mu_n, -A^* \mu_n)$ is monotonely related to the graph of A since, by Lemma 2,

$$\langle \mu_n, -A^* \mu_n \rangle = -n \cdot (-n^{-2}) \cdot n \geqslant 0.$$

But if $n \neq m$, then $(\mu_n, -A^* \mu_n)$ and $(\mu_m, -A^* \mu_m)$ are not monotonely related since, by Lemma 2,

$$\begin{aligned} \langle \mu_n - \mu_m, -A^* \mu_n + A^* \mu_m \rangle &= \langle \mu_n - \mu_m, -A^*(\mu_n - \mu_m) \rangle \\ &= -(n-m) \cdot (-n^{-2} \cdot n + m^{-2} \cdot m) < 0, \end{aligned}$$

which concludes the proof. Q.E.D.

Remark. In the situation of the above Proposition, there are infinitely many *linear* maximal monotone operators $B: D(B) \subset (l^\infty)^* \to l^\infty$ which extend A. This follows easily from our construction and from the following two simple facts: (a) in a dual pair (E, F), let $T: D(T) \subset E \to F$ be a linear monotone operator and let $(e, f) \in E \times F$ be monotonely related to the graph of T, with $e \notin D(T)$; then the linear extension of T to $\operatorname{vct}[D(T), e]$:

$$\tilde{T}(x + \lambda e) = T(x) + \lambda f \quad \text{for } x \in D(T) \text{ and } \lambda \in \mathbf{R},$$

is still monotone; (b) let $T: D(T) \subset E \to F$ be a linear monotone operator with a $\sigma(E, F)$ dense domain; if T is maximal among all linear monotone operators, then T is maximal monotone.

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