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CONTRACTIONS OF CONVEX SETS

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ABSTRACT. In this paper it is shown that, in a vector space over any ordered field, a noninfinitesimal contraction of a convex set K can be written as an intersection of translates of K.

A subset K of a vector space over a totally ordered field is called *convex* provided $\lambda x + (1 - \lambda)y$ is in K whenever x and y are in K and λ is a scalar such that $0 \leqslant \lambda \leqslant 1$. Recalling that any ordered field F has characteristic zero, and hence contains a copy of the rational numbers, we shall say that a positive element μ in F is *infinitesimal* if $\mu < r$ for all positive rational numbers r. (For further discussion and examples of ordered fields, see [1, Chapter 13].) In this note we shall prove the following intersection theorem:

THEOREM. *Suppose K is a convex set in a vector space over an ordered field and μ is a positive scalar less than* 1. *If μ is not infinitesimal, then, for some set T of vectors,*

$$\mu K = \bigcap \{K + t: t \in T\}.$$

It is easy to see that this result is plausible by considering either a square or a triangle in the plane, or in fact, any closed convex set. Difficulties arise, however, in the case of a convex set which includes only a portion of its boundary–say, the open unit disk together with the points of its circumference with rational x-coordinate.

PROOF OF THE THEOREM. Let μ be a noninfinitesimal positive scalar less than 1. Set $P = \mu K$ and suppose q is a point not in P. To prove the theorem, we must find a vector t such that $P \subseteq K + t$ but $q \notin K + t$. We distinguish two cases:

Case I. For all x in P, we have $q + \mu(x - q) \in P$. Let $t = (1 - \mu^{-1})q$ so that

$$K + t = \mu^{-1}P + (1 - \mu^{-1})q = q + \mu^{-1}(P - q).$$

The vectors in $P - q$ are all nonzero since q is not in P. Thus q is not in $K + t$. But if x is in P, then $q + \mu(x - q) \in P$ by the case hypothesis, so that $x \in q + \mu^{-1}(P - q) = K + t$. Thus $P \subseteq K + t$.

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Case II. There is a point p in P such that $q + \mu(p - q) \notin P$. Note that if $0 < \lambda < \mu$, then the segment from $q + \lambda(p - q)$ to $q + (p - q)$ contains $q + \mu(p - q)$. Since $q + (p - q) = p \in P$ and P is convex, the assumption on p forces $q + \lambda(p - q) \notin P$.

Now since μ is not infinitesimal, the positive integral powers of $1 - \mu$ become ultimately smaller than any preassigned positive rational number and, hence, smaller than any preassigned positive noninfinitesimal. Hence, $(1 - \mu)^n < \mu$ for some sufficiently large positive integer n. By the preceding note, $q + (1 - \mu)^n(p - q)$ cannot belong to P.

Let m be the smallest positive integer such that $q + (1 - \mu)^m(p - q) \notin P$, and set $v = q + (1 - \mu)^{m-1}(p - q)$. (Here $v = p$ if $m = 1$.) Then v belongs to P. If x is any point in P, then $(1 - \mu)v + \mu x \in P$ since P is convex. Whence

$$x \in \mu^{-1}(P - (1 - \mu)v) = K + (1 - \mu^{-1})v.$$

Thus letting $t = (1 - \mu^{-1})v$, we have $P \subseteq K + t$. But

$$\mu q + (1 - \mu)v = q + (1 - \mu)(v - q) = q + (1 - \mu)^m(p - q) \notin P.$$

Consequently, q does not belong to

$$\mu^{-1}(P - (1 - \mu)v) = K + (1 - \mu^{-1})v = K + t.$$

This completes the proof of the theorem.

We conclude with a simple one-dimensional example to show that the result cannot be extended to include infinitesimal contractions. Let F be an ordered field which contains a positive infinitesimal element δ [1, p. 70]. Take the convex set $K = \{x \in F: x \geqslant 0 \text{ and for some integer } n, x \leqslant n\}$. If $\lambda \in F$ such that $\delta K \subseteq K - \lambda$, then $0 \in K - \lambda$ since $0 \in \delta K$. Thus $\lambda \in K$, so $\lambda + 1$ must also be in K. Whence $1 \in K - \lambda$. Thus 1 belongs to every translate of K containing δK, but 1 does not belong to δK since δ is infinitesimal.

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