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NUMBER OF ODD BINOMIAL COEFFICIENTS

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ABSTRACT. Let $F(n)$ denote the number of odd numbers in the first n rows of Pascal's triangle, and $\theta = (\log 3)/(\log 2)$. Then $\alpha = \limsup F(n)/n^\theta = 1$, and $\beta = \liminf F(n)/n^\theta = 0.812\,556\ldots$.

It is known that almost all binomial coefficients are even numbers (see for example [1]–[3]). This means

$$\lim_{n\to\infty} F(n)\Big/\binom{n+1}{2} = \lim_{n\to\infty} F(n)/n^2 = 0,$$

if $F(n)$ denotes the number of odd numbers in the first n rows of Pascal's triangle. Recently in [4] and [5] it is asked more precisely for the asymptotic behavior of $F(n)$. Let

$$\alpha = \lim_{n\to\infty}\sup F(n)/n^\theta, \qquad \beta = \lim_{n\to\infty}\inf F(n)/n^\theta, \tag{1}$$

and

$$\theta = (\log 3)/(\log 2) = 1.584\,962\ldots. \tag{2}$$

Then it is shown in [5] that

$$1 \leqslant \alpha \leqslant 1.052, \quad \text{and} \quad 0.72 \leqslant \beta \leqslant (9/7)(3/4)^\theta \leqslant 0.815.$$

Furthermore it is conjectured that 1 and $(9/7)(3/4)^\theta = 3^\theta/7 = 0.814\,931\ldots$ are the true values of α and β. In this note we will prove $\alpha = 1$ and $\beta = 0.812\,556\ldots$.

THEOREM 1. $\alpha = 1$.

PROOF. Since

$$\binom{n}{0} = \binom{n}{n} = 1, \quad \text{and} \quad \binom{n}{i} \equiv 0 \pmod 2, \qquad 1 \leqslant i \leqslant n-1,$$

for $n = 2^r$, $r = 0, 1, \ldots$, we have the recursion

$$F(2^r + x) = F(2^r) + 2F(x), \qquad 0 \leqslant x \leqslant 2^r, \quad r = 0, 1, \ldots, \tag{3}$$

if, in addition, $F(0) = 0$ is defined. From (3), by induction on r, we get

$$F(2^r) = 3^r, \tag{4}$$

and thus $F(2^r)/2^{r\theta} = 3^r/2^{r\theta} = 1$ for all r, which yields $\alpha \geqslant 1$.

Next we assert

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(5) $$F(2^r + x)/(2^r + x)^\theta \leqslant 1 \quad \text{for} \quad 0 \leqslant x \leqslant 2^r, \quad r = 0, 1, \ldots .$$

This is true for $r = 0$. If we assume the validity of (5) for all natural numbers $\leqslant r - 1$, we can use $F(x) \leqslant x^\theta$ for $0 \leqslant x \leqslant 2^r$ to get from (3) and (4) that

$$\frac{F(2^r + x)}{(2^r + x)^\theta} = \frac{F(2^r) + 2F(x)}{(2^r + x)^\theta} \leqslant \frac{3^r + 2x^\theta}{(2^r + x)^\theta} = f(x), \qquad 0 \leqslant x \leqslant 2^r.$$

From

$$\frac{df}{dx} = \frac{\theta}{(2^r + x)^{\theta+1}}(2^{r+1}x^{\theta-1} - 3^r) = 0$$

it follows that $f(x)$ has exactly one extremum. This together with $f(0) = f(2^r) = 1$ and $f(2^{r-1}) = 5/3^\theta < 1$ yields $f(x) \leqslant 1$ for $0 \leqslant x \leqslant 2^r$. Thus (5) is proved by induction on r, and from (5) we conclude $\alpha \leqslant 1$.

Theorem 2. $\beta = 0.812\,556 \ldots .$

Proof. We consider the sequence

(6) $$\{q_r\} = \{F(n_r)/n_r^\theta\} \quad \text{with} \quad n_r = 2n_{r-1} \pm 1, \quad n_0 = 1,$$

where $+$ or $-$ is chosen so that q_r becomes minimal. So for $r = 1, 2, \ldots, 25$ we have to choose

(7) $$+ - + - + + - + - + + - + - + - + + - + - + + - + .$$

If t_r denotes the sum of the binary digits of n_r, the first eleven values of n_r, $F(n_r)$, and t_r are

r	n_r	$F(n_r)$	t_r
0	1	1	1
1	3	5	2
2	5	11	2
3	11	37	3
4	21	103	3
5	43	317	4
6	87	967	5
7	173	2 869	5
8	347	8 639	6
9	693	25 853	6
10	1 387	77 623	7

Lemma. *$\{q_r\}$ is strictly decreasing.*

Proof. We suppose

(8) $$F(2n_r + 1)/(2n_r + 1)^\theta \geqslant q_r \quad \text{and} \quad F(2n_r - 1)/(2n_r - 1)^\theta \geqslant q_r.$$

Using (3), (4), and the binary representation of n_r we obtain

(9) $$F(2n_r \pm 1) = 3F(n_r) \pm 2^{t_r}, \qquad t_r = t_{r-1} + \tfrac{1}{2} \pm \tfrac{1}{2}.$$

(Here the reader may recognize the well-known result (see [5] for references) that the number of odd $\binom{n}{i}$ is 2^t, where t is the number of binary digits of n.) We insert (9) and (6) in (8), and substitute $2n_r = a$ and $2^{t_r}/(3F(n_r)) = b$ to get

$$1 + b \geqslant \left(1 + \frac{1}{a}\right)^\theta = 1 + \frac{\theta}{a} + \frac{\theta(\theta-1)}{2a^2}$$
$$+\theta(\theta-1)\sum_{i=1}^{\infty}\left(\frac{-1}{a}\right)^{i+2}\frac{(2-\theta)\cdots(i+1-\theta)}{(i+2)!},$$
$$1 - b \geqslant \left(1 - \frac{1}{a}\right)^\theta = 1 - \frac{\theta}{a} + \frac{\theta(\theta-1)}{2a^2}$$
$$+\theta(\theta-1)\sum_{i=1}^{\infty}\left(\frac{1}{a}\right)^{i+2}\frac{(2-\theta)\cdots(i+1-\theta)}{(i+2)!}.$$

Addition of the last two inequalities yields the contradiction

$$2 \geqslant 2 + \theta(\theta-1)/a^2 + \cdots > 2.$$

Thus the inequalities (8) cannot both be true, which proves the Lemma.

Now $q_r > 0$ together with the Lemma proves the convergence of $\{q_r\}$. It follows that

$$B \leqslant q = \lim_{r\to\infty} q_r < q_{19} = 0.812\,556\ldots, \tag{10}$$

with

$$n_{19} = 710\,317$$
$$= 2^{19} + 2^{17} + 2^{15} + 2^{14} + 2^{12} + 2^{10} + 2^9 + 2^7 + 2^5 + 2^3 + 2^2 + 1.$$

We still have to prove

$$F(n)/n^\theta > 0.812\,556 = \gamma. \tag{11}$$

This is true for $1 \leqslant n \leqslant 2$, and we assume the validity of (11) for $1 \leqslant n \leqslant 2^r$. To obtain the step from r to $r+1$ in a proof of (11) by induction on r we have to conclude from this assumption that (11) also holds for $n = 2^r + x$, $1 \leqslant x \leqslant 2^r$. We divide this interval into eleven intervals:

$$n = 2^{r-s}m + x, \qquad 1 \leqslant x \leqslant 2^{r-s},$$
$$m = n_s \quad \text{for} \quad s = 1, 3, 6, 8, 10,$$
$$m = n_s - 1 \quad \text{for} \quad s = 2, 4, 5, 7, 9, 10.$$

Let t be the sum of the binary digits of m, and $2^s < m < 2^{s+1}$. Then for $1 \leqslant x \leqslant 2^{r-s}$ we get from (3) and (4) that

$$\frac{F(2^{r-s}m + x)}{(2^{r-s}m + x)^\theta} = \frac{3^{r-s}F(m) + 2^tF(x)}{(2^{r-s}m + x)^\theta} > \frac{3^{r-s}F(m) + 2^t\gamma x^\theta}{(2^{r-s}m + x)^\theta} = f_s(x). \tag{12}$$

The unique extremum of $f_s(x)$ is a minimum at

$$x_{\min} = 2^{r-s}\left(F(m)/\gamma m 2^t\right)^{1/(\theta-1)}.$$

For $m = n_s$ and $s = 1, 3, 6, 8, 10$ we check by calculation that

$$(13)\quad f_s(x) \geqslant f_s(x_{\min}) = \left(\left(F(m)/m^\theta\right)^{1/(1-\theta)} + \left(\gamma 2^t\right)^{1/(1-\theta)}\right)^{1-\theta} > \gamma$$

is fulfilled. For $m = n_s - 1$ and $s = 2, 4, 5, 7, 9, 10$ we ascertain that in these cases $x_{\min} > 2^{r-s}$. Then for $s \neq 10$,

$$f_s(x) \geqslant f_s(2^{r-s}) = \frac{F(n_s - 1) + \gamma 2^{t_s - 1}}{n_s^\theta} = \frac{F(n_s) - (1-\gamma)2^{t_s-1}}{n_s^\theta} > \gamma$$

is seen to be true by calculation. In the case $m = n_{10} - 1$, $s = 10$, we first have

$$f_{10}(x) \geqslant f_{10}(2^{r-11}) = \frac{3F(n_{10}) - (3-\gamma)2^{t_{10}-1}}{(2n_{10}-1)^\theta} > \gamma, \qquad 1 \leqslant x \leqslant 2^{r-11}.$$

For the remaining partial interval

$$n = 2^{r-10}(n_{10} - 1) + 2^{r-11} + x = 2^{r-11}(2n_{10} - 1) + x, \qquad 1 \leqslant x \leqslant 2^{r-11},$$

we choose $m = 2n_{10} - 1$ and $s = 11$ in (12), and check the validity of (13).

Now the induction on r is complete, and we have proved (11) for all n. Inequalities (10) and (11) then yield Theorem 2.

At the end we remark that q from (10) probably will be the exact value of β. Moreover, we conjecture for all r,

$$F(n)/n^\theta \geqslant q_r \quad \text{for} \quad 2^r \leqslant n \leqslant 2^{r+1}.$$

It seems, however, that for a general proof we should know some more properties of the sequence of plus and minus signs beginning with (7). Are there any regularities in this sequence?

References

1. N. J. Fine, *Binomial coefficients modulo a prime*, Amer. Math. Monthly **54** (1947), 589–592. MR **9**, 331.

2. H. Harborth, *Über die Teilbarkeit im Pascal-Dreieck*, Math.-Phys. Semesterber. **22** (1975), 13–21.

3. D. Singmaster, *Notes on binomial coefficients*. III: *Any integer divides almost all binomial coefficients*, J. London Math. Soc. (2) **8** (1974), 555–560.

4. K. B. Stolarsky, *Digital sums and binomial coefficients*, Notices Amer. Math. Soc. **22** (1975), A-669. Abstract #728-A7.

5. ______, *Power and exponential sums of digital sums related to binomial coefficient parity*, SIAM J. Appl. Math. (to appear).

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