

## NUMBER OF ODD BINOMIAL COEFFICIENTS

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**ABSTRACT.** Let  $F(n)$  denote the number of odd numbers in the first  $n$  rows of Pascal's triangle, and  $\theta = (\log 3)/(\log 2)$ . Then  $\alpha = \limsup F(n)/n^\theta = 1$ , and  $\beta = \liminf F(n)/n^\theta = 0.812\ 556\dots$

It is known that almost all binomial coefficients are even numbers (see for example [1]–[3]). This means

$$\lim_{n \rightarrow \infty} F(n) / \binom{n+1}{2} = \lim_{n \rightarrow \infty} F(n)/n^2 = 0,$$

if  $F(n)$  denotes the number of odd numbers in the first  $n$  rows of Pascal's triangle. Recently in [4] and [5] it is asked more precisely for the asymptotic behavior of  $F(n)$ . Let

$$(1) \quad \alpha = \lim_{n \rightarrow \infty} \sup F(n)/n^\theta, \quad \beta = \lim_{n \rightarrow \infty} \inf F(n)/n^\theta,$$

and

$$(2) \quad \theta = (\log 3)/(\log 2) = 1.584\ 962\dots$$

Then it is shown in [5] that

$$1 < \alpha \leq 1.052, \quad \text{and} \quad 0.72 \leq \beta \leq (9/7)(3/4)^\theta \leq 0.815.$$

Furthermore it is conjectured that 1 and  $(9/7)(3/4)^\theta = 3^\theta/7 = 0.814\ 931\dots$  are the true values of  $\alpha$  and  $\beta$ . In this note we will prove  $\alpha = 1$  and  $\beta = 0.812\ 556\dots$

**THEOREM 1.**  $\alpha = 1$ .

**PROOF.** Since

$$\binom{n}{0} = \binom{n}{n} = 1, \quad \text{and} \quad \binom{n}{i} \equiv 0 \pmod{2}, \quad 1 \leq i \leq n-1,$$

for  $n = 2^r$ ,  $r = 0, 1, \dots$ , we have the recursion

$$(3) \quad F(2^r + x) = F(2^r) + 2F(x), \quad 0 \leq x < 2^r, \quad r = 0, 1, \dots,$$

if, in addition,  $F(0) = 0$  is defined. From (3), by induction on  $r$ , we get

$$(4) \quad F(2^r) = 3^r,$$

and thus  $F(2^r)/2^{r\theta} = 3^r/2^{r\theta} = 1$  for all  $r$ , which yields  $\alpha \geq 1$ .

Next we assert

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$$(5) \quad F(2^r + x)/(2^r + x)^\theta \leq 1 \quad \text{for } 0 \leq x \leq 2^r, \quad r = 0, 1, \dots$$

This is true for  $r = 0$ . If we assume the validity of (5) for all natural numbers  $\leq r - 1$ , we can use  $F(x) \leq x^\theta$  for  $0 \leq x \leq 2^r$  to get from (3) and (4) that

$$\frac{F(2^r + x)}{(2^r + x)^\theta} = \frac{F(2^r) + 2F(x)}{(2^r + x)^\theta} \leq \frac{3^r + 2x^\theta}{(2^r + x)^\theta} = f(x), \quad 0 \leq x \leq 2^r.$$

From

$$\frac{df}{dx} = \frac{\theta}{(2^r + x)^{\theta+1}} (2^{r+1}x^{\theta-1} - 3^r) = 0$$

it follows that  $f(x)$  has exactly one extremum. This together with  $f(0) = f(2^r) = 1$  and  $f(2^{r-1}) = 5/3^\theta < 1$  yields  $f(x) \leq 1$  for  $0 \leq x \leq 2^r$ . Thus (5) is proved by induction on  $r$ , and from (5) we conclude  $\alpha \leq 1$ .

THEOREM 2.  $\beta = 0.812\,556\, \dots$

PROOF. We consider the sequence

$$(6) \quad \{q_r\} = \{F(n_r)/n_r^\theta\} \quad \text{with } n_r = 2n_{r-1} \pm 1, \quad n_0 = 1,$$

where  $+$  or  $-$  is chosen so that  $q_r$  becomes minimal. So for  $r = 1, 2, \dots, 25$  we have to choose

$$(7) \quad + - + - + + - + - - + + - + - + - + + - + - + + - +.$$

If  $t_r$  denotes the sum of the binary digits of  $n_r$ , the first eleven values of  $n_r$ ,  $F(n_r)$ , and  $t_r$  are

$r$	$n_r$	$F(n_r)$	$t_r$
0	1	1	1
1	3	5	2
2	5	11	2
3	11	37	3
4	21	103	3
5	43	317	4
6	87	967	5
7	173	2 869	5
8	347	8 639	6
9	693	25 853	6
10	1 387	77 623	7

LEMMA.  $\{q_r\}$  is strictly decreasing.

PROOF. We suppose

$$(8) \quad F(2n_r + 1)/(2n_r + 1)^\theta \geq q_r \quad \text{and} \quad F(2n_r - 1)/(2n_r - 1)^\theta \geq q_r.$$

Using (3), (4), and the binary representation of  $n_r$  we obtain

$$(9) \quad F(2n_r \pm 1) = 3F(n_r) \pm 2^{t_r}, \quad t_r = t_{r-1} + \frac{1}{2} \pm \frac{1}{2}.$$

(Here the reader may recognize the well-known result (see [5] for references) that the number of odd  $\binom{n}{i}$  is  $2^t$ , where  $t$  is the number of binary digits of  $n$ .) We insert (9) and (6) in (8), and substitute  $2n_r = a$  and  $2^r/(3F(n_r)) = b$  to get

$$\begin{aligned}
 1 + b &\geq \left(1 + \frac{1}{a}\right)^\theta = 1 + \frac{\theta}{a} + \frac{\theta(\theta - 1)}{2a^2} \\
 &\quad + \theta(\theta - 1) \sum_{i=1}^{\infty} \left(\frac{-1}{a}\right)^{i+2} \frac{(2 - \theta) \cdots (i + 1 - \theta)}{(i + 2)!}, \\
 1 - b &\geq \left(1 - \frac{1}{a}\right)^\theta = 1 - \frac{\theta}{a} + \frac{\theta(\theta - 1)}{2a^2} \\
 &\quad + \theta(\theta - 1) \sum_{i=1}^{\infty} \left(\frac{1}{a}\right)^{i+2} \frac{(2 - \theta) \cdots (i + 1 - \theta)}{(i + 2)!}.
 \end{aligned}$$

Addition of the last two inequalities yields the contradiction

$$2 \geq 2 + \theta(\theta - 1)/a^2 + \cdots > 2.$$

Thus the inequalities (8) cannot both be true, which proves the Lemma.

Now  $q_r > 0$  together with the Lemma proves the convergence of  $\{q_r\}$ . It follows that

$$(10) \quad B \leq q = \lim_{r \rightarrow \infty} q_r < q_{19} = 0.812\ 556 \dots,$$

with

$$\begin{aligned}
 n_{19} &= 710\ 317 \\
 &= 2^{19} + 2^{17} + 2^{15} + 2^{14} + 2^{12} + 2^{10} + 2^9 + 2^7 + 2^5 + 2^3 + 2^2 + 1.
 \end{aligned}$$

We still have to prove

$$(11) \quad F(n)/n^\theta > 0.812\ 556 = \gamma.$$

This is true for  $1 \leq n \leq 2$ , and we assume the validity of (11) for  $1 \leq n \leq 2^r$ . To obtain the step from  $r$  to  $r + 1$  in a proof of (11) by induction on  $r$  we have to conclude from this assumption that (11) also holds for  $n = 2^r + x$ ,  $1 \leq x \leq 2^r$ . We divide this interval into eleven intervals:

$$\begin{aligned}
 n &= 2^{r-s}m + x, & 1 \leq x \leq 2^{r-s}, \\
 m &= n_s & \text{for } s = 1, 3, 6, 8, 10, \\
 m &= n_s - 1 & \text{for } s = 2, 4, 5, 7, 9, 10.
 \end{aligned}$$

Let  $t$  be the sum of the binary digits of  $m$ , and  $2^s < m < 2^{s+1}$ . Then for  $1 \leq x \leq 2^{r-s}$  we get from (3) and (4) that

$$(12) \quad \frac{F(2^{r-s}m + x)}{(2^{r-s}m + x)^\theta} = \frac{3^{r-s}F(m) + 2^rF(x)}{(2^{r-s}m + x)^\theta} > \frac{3^{r-s}F(m) + 2^r\gamma x^\theta}{(2^{r-s}m + x)^\theta} = f_s(x).$$

The unique extremum of  $f_s(x)$  is a minimum at

$$x_{\min} = 2^{r-s} (F(m)/\gamma m 2^r)^{1/(\theta-1)}.$$

For  $m = n_s$  and  $s = 1, 3, 6, 8, 10$  we check by calculation that

$$(13) \quad f_s(x) \geq f_s(x_{\min}) = \left( (F(m)/m^\theta)^{1/(1-\theta)} + (\gamma 2^t)^{1/(1-\theta)} \right)^{1-\theta} > \gamma$$

is fulfilled. For  $m = n_s - 1$  and  $s = 2, 4, 5, 7, 9, 10$  we ascertain that in these cases  $x_{\min} > 2^{r-s}$ . Then for  $s \neq 10$ ,

$$f_s(x) \geq f_s(2^{r-s}) = \frac{F(n_s - 1) + \gamma 2^{t-1}}{n_s^\theta} = \frac{F(n_s) - (1 - \gamma) 2^{t-1}}{n_s^\theta} > \gamma$$

is seen to be true by calculation. In the case  $m = n_{10} - 1$ ,  $s = 10$ , we first have

$$f_{10}(x) \geq f_{10}(2^{r-11}) = \frac{3F(n_{10}) - (3 - \gamma) 2^{t_{10}-1}}{(2n_{10} - 1)^\theta} > \gamma, \quad 1 \leq x \leq 2^{r-11}.$$

For the remaining partial interval

$$n = 2^{r-10}(n_{10} - 1) + 2^{r-11} + x = 2^{r-11}(2n_{10} - 1) + x, \quad 1 \leq x \leq 2^{r-11},$$

we choose  $m = 2n_{10} - 1$  and  $s = 11$  in (12), and check the validity of (13).

Now the induction on  $r$  is complete, and we have proved (11) for all  $n$ . Inequalities (10) and (11) then yield Theorem 2.

At the end we remark that  $q$  from (10) probably will be the exact value of  $\beta$ . Moreover, we conjecture for all  $r$ ,

$$F(n)/n^\theta \geq q_r \quad \text{for } 2^r \leq n \leq 2^{r+1}.$$

It seems, however, that for a general proof we should know some more properties of the sequence of plus and minus signs beginning with (7). Are there any regularities in this sequence?

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