NUMBER OF ODD BINOMIAL COEFFICIENTS

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Abstract. Let \( F(n) \) denote the number of odd numbers in the first \( n \) rows of Pascal's triangle, and \( \theta = (\log 3)/(\log 2) \). Then \( \alpha = \limsup F(n)/n^{\theta} = 1 \), and \( \beta = \liminf F(n)/n^{\theta} = 0.812 \ldots \).

It is known that almost all binomial coefficients are even numbers (see for example [1]–[3]). This means

\[
\lim_{n \to \infty} \frac{F(n)}{n^{\theta}} = \frac{n + 1}{2} = \lim_{n \to \infty} \frac{F(n)}{n^2} = 0,
\]

if \( F(n) \) denotes the number of odd numbers in the first \( n \) rows of Pascal's triangle. Recently in [4] and [5] it is asked more precisely for the asymptotic behavior of \( F(n) \). Let

\[
\alpha = \limsup_{n \to \infty} \frac{F(n)}{n^{\theta}}, \quad \beta = \liminf_{n \to \infty} \frac{F(n)}{n^{\theta}},
\]

and

\[
\theta = (\log 3)/(\log 2) = 1.584 \ldots.
\]

Then it is shown in [5] that

\[
1 < \alpha \leq 1.052, \quad 0.72 < \beta < (9/7)(3/4)^{\theta} \leq 0.815.
\]

Furthermore it is conjectured that 1 and \((9/7)(3/4)^{\theta} = 3^{\theta}/7 = 0.814 \ldots\) are the true values of \( \alpha \) and \( \beta \). In this note we will prove \( \alpha = 1 \) and \( \beta = 0.812 \ldots \).

Theorem 1. \( \alpha = 1 \).

Proof. Since

\[
\binom{n}{0} = \binom{n}{n} = 1, \quad \text{and} \quad \binom{n}{i} \equiv 0 \pmod{2}, \quad 1 < i < n - 1,
\]

for \( n = 2^r, r = 0, 1, \ldots \), we have the recursion

\[
F(2^r + x) = F(2^r) + 2F(x), \quad 0 < x < 2^r, \quad r = 0, 1, \ldots,
\]

if, in addition, \( F(0) = 0 \) is defined. From (3), by induction on \( r \), we get

\[
F(2^r) = 3^r,
\]

and thus \( F(2^r)/2^{r\theta} = 3^r/2^{\theta} = 1 \) for all \( r \), which yields \( \alpha > 1 \).

Next we assert...
(5) \[ \frac{F(2r + x)}{(2r + x)^\theta} \leq 1 \quad \text{for} \quad 0 \leq x \leq 2r, \quad r = 0, 1, \ldots. \]

This is true for \( r = 0 \). If we assume the validity of (5) for all natural numbers \( < r - 1 \), we can use \( F(x) \leq x^\theta \) for \( 0 \leq x \leq 2r \) to get from (3) and (4) that

\[
\frac{F(2r + x)}{(2r + x)^\theta} = \frac{F(2r) + 2F(x)}{(2r + x)^\theta} \leq \frac{3r + 2x^\theta}{(2r + x)^\theta} = f(x), \quad 0 \leq x \leq 2r.
\]

From

\[
\frac{df}{dx} = \frac{\theta}{(2r + x)^{\theta+1}} (2r^\theta + x^{\theta-1} - x) = 0
\]

it follows that \( f(x) \) has exactly one extremum. This together with \( f(0) = f(2r) = 1 \) and \( f(2r-1) = 5/3^\theta < 1 \) yields \( f(x) \leq 1 \) for \( 0 \leq x \leq 2r \). Thus (5) is proved by induction on \( r \), and from (5) we conclude \( \alpha \leq 1 \).

**Theorem 2.** \( \beta = 0.812 \ 556 \ldots \)

**Proof.** We consider the sequence

(6) \( \{ q_r \} = \{ F(n_r)/n_r^\theta \} \) with \( n_r = 2n_{r-1} \pm 1, \quad n_0 = 1 \),

where + or - is chosen so that \( q_r \) becomes minimal. So for \( r = 1, 2, \ldots, 25 \) we have to choose

(7) \( + - + + + - + - - + + - - + + + - - - + + + - + - + - + . \)

If \( t_r \) denotes the sum of the binary digits of \( n_r \), the first eleven values of \( n_r, F(n_r), \) and \( t_r \) are

<table>
<thead>
<tr>
<th>( r )</th>
<th>( n_r )</th>
<th>( F(n_r) )</th>
<th>( t_r )</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>11</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>37</td>
<td>3</td>
</tr>
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<td>967</td>
<td>5</td>
</tr>
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<td>173</td>
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<td>8 639</td>
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</tr>
<tr>
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<td>693</td>
<td>25 853</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>1 387</td>
<td>77 623</td>
<td>7</td>
</tr>
</tbody>
</table>

**Lemma.** \( \{ q_r \} \) is strictly decreasing.

**Proof.** We suppose

(8) \( F(2n_r + 1)/(2n_r + 1)^\theta \geq q_r \) and \( F(2n_r - 1)/(2n_r - 1)^\theta \geq q_r \).

Using (3), (4), and the binary representation of \( n_r \), we obtain

(9) \[ F(2n_r \pm 1) = 3F(n_r) \pm 2t_r, \quad t_r = t_{r-1} + \frac{1}{2} \pm \frac{1}{2}. \]
(Here the reader may recognize the well-known result (see [5] for references) that the number of odd \( \binom{n}{2t} \) is \( 2^t \), where \( t \) is the number of binary digits of \( n \).)

We insert (9) and (6) in (8), and substitute \( 2n_r = a \) and \( 2^r/(3F(n_r)) = b \) to get

\[
1 + b > \left( 1 + \frac{1}{a} \right)^\theta = 1 + \frac{\theta}{a} + \frac{\theta(\theta - 1)}{2a^2} + \theta(\theta - 1) \sum_{i=1}^{\infty} \left( \frac{-1}{a} \right)^{i+2} \frac{(2 - \theta) \cdots (i + 1 - \theta)}{(i + 2)!}.
\]

\[
1 - b > \left( 1 - \frac{1}{a} \right)^\theta = 1 - \frac{\theta}{a} + \frac{\theta(\theta - 1)}{2a^2} + \theta(\theta - 1) \sum_{i=1}^{\infty} \left( \frac{1}{a} \right)^{i+2} \frac{(2 - \theta) \cdots (i + 1 - \theta)}{(i + 2)!}.
\]

Addition of the last two inequalities yields the contradiction

\[
2 > 2 + \theta(\theta - 1)/a^2 + \cdots > 2.
\]

Thus the inequalities (8) cannot both be true, which proves the Lemma.

Now \( q_r > 0 \) together with the Lemma proves the convergence of \( \{q_r\} \). It follows that

\[
(10) \quad B < q = \lim_{r \to \infty} q_r < q_{19} = 0.812556 \ldots,
\]

with

\[
n_{19} = 710317
\]

\[
= 2^{19} + 2^{17} + 2^{15} + 2^{14} + 2^{12} + 2^{10} + 2^9 + 2^7 + 2^5 + 2^3 + 2^2 + 1.
\]

We still have to prove

\[
(11) \quad F(n)/n^\theta > 0.812556 = \gamma.
\]

This is true for \( 1 < n < 2 \), and we assume the validity of (11) for \( 1 < n < 2^t \).

To obtain the step from \( r \) to \( r + 1 \) in a proof of (11) by induction on \( r \) we have to conclude from this assumption that (11) also holds for \( n = 2^r + x \), \( 1 < x < 2^r \). We divide this interval into eleven intervals:

\[
n = 2^{-s}m + x, \quad 1 < x < 2^{-s};
\]

\[
m = n_s \quad \text{for} \quad s = 1, 3, 6, 8, 10,
\]

\[
m = n_s - 1 \quad \text{for} \quad s = 2, 4, 5, 7, 9, 10.
\]

Let \( t \) be the sum of the binary digits of \( m \), and \( 2^t < m < 2^{t+1} \). Then for \( 1 < x < 2^{-s-1} \) we get from (3) and (4) that

\[
(12) \quad \frac{F(2^{-s}m + x)}{(2^{-s}m + x)^\theta} = \frac{3^{-s}F(m) + 2^tF(x)}{(2^{-s}m + x)^\theta} > \frac{3^{-s}F(m) + 2^t\gamma x^\theta}{(2^{-s}m + x)^\theta} = f_s(x).
\]

The unique extremum of \( f_s(x) \) is a minimum at

\[
x_{\text{min}} = 2^{-s}\left( F(m)/\gamma m2^t \right)^{1/(\theta - 1)}.
\]
For $m = n_s$ and $s = 1, 3, 6, 8, 10$ we check by calculation that

$$f_s(x) > f_s(x_{\text{min}}) = \left(\left(\frac{F(m)}{m^s}\right)^{1/(1-\theta)} + \left(\gamma 2^{i/(1-\theta)}\right)^{1-\theta}\right) > \gamma$$

is fulfilled. For $m = n_s - 1$ and $s = 2, 4, 5, 7, 9, 10$ we ascertain that in these cases $x_{\text{min}} > 2^{s-1}$. Then for $5 \leq i = 10,$

$$f_s(x) \geq f_s(2^{s-1}) = \frac{F(n_s - 1) + \gamma 2^{s-1}}{n_s^\theta} = \frac{F(n_s) - (1 - \gamma) 2^{s-1}}{n_s^\theta} > \gamma$$

is seen to be true by calculation. In the case $m = n_{10} - 1, s = 10$, we first have

$$f_{10}(x) \geq f_{10}(2^{s-1}) = \frac{3F(n_{10}) - (3 - \gamma) 2^{s-1}}{(2n_{10} - 1)^\theta} > \gamma, \quad 1 < x < 2^{s-1}.$$  

For the remaining partial interval

$$n = 2^{s-10}(n_{10} - 1) + 2^{s-11} + x = 2^{s-11}(2n_{10} - 1) + x, \quad 1 < x < 2^{s-11},$$

we choose $m = 2n_{10} - 1$ and $s = 11$ in (12), and check the validity of (13).

Now the induction on $r$ is complete, and we have proved (11) for all $n$. Inequalities (10) and (11) then yield Theorem 2.

At the end we remark that $q$ from (10) probably will be the exact value of $\beta$. Moreover, we conjecture for all $r$,

$$F(n)/n^\theta > q, \quad \text{for} \quad 2^r < n < 2^{r+1}.$$  

It seems, however, that for a general proof we should know some more properties of the sequence of plus and minus signs beginning with (7). Are there any regularities in this sequence?

References


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