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NUMBER OF ODD BINOMIAL COEFFICIENTS

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ABSTRACT. Let $F(n)$ denote the number of odd numbers in the first n rows of Pascal's triangle, and $\theta = (\log 3)/(\log 2)$. Then $\alpha = \limsup F(n)/n^\theta = 1$, and $\beta = \liminf F(n)/n^\theta = 0.812\,556\ldots$.

It is known that almost all binomial coefficients are even numbers (see for example [1]–[3]). This means

$$\lim_{n\to\infty} F(n)\Big/\binom{n+1}{2} = \lim_{n\to\infty} F(n)/n^2 = 0,$$

if $F(n)$ denotes the number of odd numbers in the first n rows of Pascal's triangle. Recently in [4] and [5] it is asked more precisely for the asymptotic behavior of $F(n)$. Let

$$\alpha = \lim_{n\to\infty} \sup F(n)/n^\theta, \qquad \beta = \lim_{n\to\infty} \inf F(n)/n^\theta, \tag{1}$$

and

$$\theta = (\log 3)/(\log 2) = 1.584\,962\ldots. \tag{2}$$

Then it is shown in [5] that

$$1 \leqslant \alpha \leqslant 1.052, \quad \text{and} \quad 0.72 \leqslant \beta \leqslant (9/7)(3/4)^\theta \leqslant 0.815.$$

Furthermore it is conjectured that 1 and $(9/7)(3/4)^\theta = 3^\theta/7 = 0.814\,931\ldots$ are the true values of α and β. In this note we will prove $\alpha = 1$ and $\beta = 0.812\,556\ldots$.

THEOREM 1. $\alpha = 1$.

PROOF. Since

$$\binom{n}{0} = \binom{n}{n} = 1, \quad \text{and} \quad \binom{n}{i} \equiv 0 \pmod 2, \qquad 1 \leqslant i \leqslant n-1,$$

for $n = 2^r$, $r = 0, 1, \ldots$, we have the recursion

$$F(2^r + x) = F(2^r) + 2F(x), \qquad 0 \leqslant x \leqslant 2^r, \quad r = 0, 1, \ldots, \tag{3}$$

if, in addition, $F(0) = 0$ is defined. From (3), by induction on r, we get

$$F(2^r) = 3^r, \tag{4}$$

and thus $F(2^r)/2^{r\theta} = 3^r/2^{r\theta} = 1$ for all r, which yields $\alpha \geqslant 1$.

Next we assert

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(5) $F(2^r + x)/(2^r + x)^\theta \leqslant 1 \quad \text{for} \quad 0 \leqslant x \leqslant 2^r, \quad r = 0, 1, \ldots.$

This is true for $r = 0$. If we assume the validity of (5) for all natural numbers $\leqslant r - 1$, we can use $F(x) \leqslant x^\theta$ for $0 \leqslant x \leqslant 2^r$ to get from (3) and (4) that

$$\frac{F(2^r + x)}{(2^r + x)^\theta} = \frac{F(2^r) + 2F(x)}{(2^r + x)^\theta} \leqslant \frac{3^r + 2x^\theta}{(2^r + x)^\theta} = f(x), \qquad 0 \leqslant x \leqslant 2^r.$$

From

$$\frac{df}{dx} = \frac{\theta}{(2^r + x)^{\theta+1}}(2^{r+1}x^{\theta-1} - 3^r) = 0$$

it follows that $f(x)$ has exactly one extremum. This together with $f(0) = f(2^r) = 1$ and $f(2^{r-1}) = 5/3^\theta < 1$ yields $f(x) \leqslant 1$ for $0 \leqslant x \leqslant 2^r$. Thus (5) is proved by induction on r, and from (5) we conclude $\alpha \leqslant 1$.

Theorem 2. $\beta = 0.812\,556\ldots.$

Proof. We consider the sequence

(6) $\{q_r\} = \{F(n_r)/n_r^\theta\} \quad \text{with} \quad n_r = 2n_{r-1} \pm 1, \quad n_0 = 1,$

where $+$ or $-$ is chosen so that q_r becomes minimal. So for $r = 1, 2, \ldots, 25$ we have to choose

(7) $+ - + - + + - + - + + - + - + - + + - + - + + - +.$

If t_r denotes the sum of the binary digits of n_r, the first eleven values of n_r, $F(n_r)$, and t_r are

r	n_r	$F(n_r)$	t_r
0	1	1	1
1	3	5	2
2	5	11	2
3	11	37	3
4	21	103	3
5	43	317	4
6	87	967	5
7	173	2 869	5
8	347	8 639	6
9	693	25 853	6
10	1 387	77 623	7

Lemma. *$\{q_r\}$ is strictly decreasing.*

Proof. We suppose

(8) $F(2n_r + 1)/(2n_r + 1)^\theta \geqslant q_r \quad \text{and} \quad F(2n_r - 1)/(2n_r - 1)^\theta \geqslant q_r.$

Using (3), (4), and the binary representation of n_r we obtain

(9) $F(2n_r \pm 1) = 3F(n_r) \pm 2^{t_r}, \qquad t_r = t_{r-1} + \tfrac{1}{2} \pm \tfrac{1}{2}.$

(Here the reader may recognize the well-known result (see [5] for references) that the number of odd $\binom{n}{i}$ is 2^t, where t is the number of binary digits of n.) We insert (9) and (6) in (8), and substitute $2n_r = a$ and $2^{t_r}/(3F(n_r)) = b$ to get

$$1 + b \geqslant \left(1 + \frac{1}{a}\right)^\theta = 1 + \frac{\theta}{a} + \frac{\theta(\theta - 1)}{2a^2}$$
$$+\theta(\theta - 1)\sum_{i=1}^{\infty}\left(\frac{-1}{a}\right)^{i+2}\frac{(2-\theta)\cdots(i+1-\theta)}{(i+2)!},$$
$$1 - b \geqslant \left(1 - \frac{1}{a}\right)^\theta = 1 - \frac{\theta}{a} + \frac{\theta(\theta - 1)}{2a^2}$$
$$+\theta(\theta - 1)\sum_{i=1}^{\infty}\left(\frac{1}{a}\right)^{i+2}\frac{(2-\theta)\cdots(i+1-\theta)}{(i+2)!}.$$

Addition of the last two inequalities yields the contradiction

$$2 \geqslant 2 + \theta(\theta - 1)/a^2 + \cdots > 2.$$

Thus the inequalities (8) cannot both be true, which proves the Lemma.

Now $q_r > 0$ together with the Lemma proves the convergence of $\{q_r\}$. It follows that

$$B \leqslant q = \lim_{r\to\infty} q_r < q_{19} = 0.812\,556\ldots, \tag{10}$$

with

$$n_{19} = 710\,317$$
$$= 2^{19} + 2^{17} + 2^{15} + 2^{14} + 2^{12} + 2^{10} + 2^9 + 2^7 + 2^5 + 2^3 + 2^2 + 1.$$

We still have to prove

$$F(n)/n^\theta > 0.812\,556 = \gamma. \tag{11}$$

This is true for $1 \leqslant n \leqslant 2$, and we assume the validity of (11) for $1 \leqslant n \leqslant 2^r$. To obtain the step from r to $r+1$ in a proof of (11) by induction on r we have to conclude from this assumption that (11) also holds for $n = 2^r + x$, $1 \leqslant x \leqslant 2^r$. We divide this interval into eleven intervals:

$$n = 2^{r-s}m + x, \qquad 1 \leqslant x \leqslant 2^{r-s},$$
$$m = n_s \quad \text{for} \quad s = 1, 3, 6, 8, 10,$$
$$m = n_s - 1 \quad \text{for} \quad s = 2, 4, 5, 7, 9, 10.$$

Let t be the sum of the binary digits of m, and $2^s < m < 2^{s+1}$. Then for $1 \leqslant x \leqslant 2^{r-s}$ we get from (3) and (4) that

$$\frac{F(2^{r-s}m + x)}{(2^{r-s}m + x)^\theta} = \frac{3^{r-s}F(m) + 2^tF(x)}{(2^{r-s}m + x)^\theta} > \frac{3^{r-s}F(m) + 2^t\gamma x^\theta}{(2^{r-s}m + x)^\theta} = f_s(x). \tag{12}$$

The unique extremum of $f_s(x)$ is a minimum at

$$x_{\min} = 2^{r-s}\left(F(m)/\gamma m 2^t\right)^{1/(\theta-1)}.$$

For $m = n_s$ and $s = 1, 3, 6, 8, 10$ we check by calculation that

$$(13) \quad f_s(x) \geqslant f_s(x_{\min}) = \left(\left(F(m)/m^\theta\right)^{1/(1-\theta)} + \left(\gamma 2^t\right)^{1/(1-\theta)}\right)^{1-\theta} > \gamma$$

is fulfilled. For $m = n_s - 1$ and $s = 2, 4, 5, 7, 9, 10$ we ascertain that in these cases $x_{\min} > 2^{r-s}$. Then for $s \neq 10$,

$$f_s(x) \geqslant f_s(2^{r-s}) = \frac{F(n_s - 1) + \gamma 2^{t_s - 1}}{n_s^\theta} = \frac{F(n_s) - (1-\gamma)2^{t_s-1}}{n_s^\theta} > \gamma$$

is seen to be true by calculation. In the case $m = n_{10} - 1$, $s = 10$, we first have

$$f_{10}(x) \geqslant f_{10}(2^{r-11}) = \frac{3F(n_{10}) - (3-\gamma)2^{t_{10}-1}}{(2n_{10}-1)^\theta} > \gamma, \qquad 1 \leqslant x \leqslant 2^{r-11}.$$

For the remaining partial interval

$$n = 2^{r-10}(n_{10} - 1) + 2^{r-11} + x = 2^{r-11}(2n_{10} - 1) + x, \qquad 1 \leqslant x \leqslant 2^{r-11},$$

we choose $m = 2n_{10} - 1$ and $s = 11$ in (12), and check the validity of (13).

Now the induction on r is complete, and we have proved (11) for all n. Inequalities (10) and (11) then yield Theorem 2.

At the end we remark that q from (10) probably will be the exact value of β. Moreover, we conjecture for all r,

$$F(n)/n^\theta \geqslant q_r \quad \text{for} \quad 2^r \leqslant n \leqslant 2^{r+1}.$$

It seems, however, that for a general proof we should know some more properties of the sequence of plus and minus signs beginning with (7). Are there any regularities in this sequence?

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