

INJECTIVE COGENERATOR RINGS AND A THEOREM OF TACHIKAWA. II

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ABSTRACT. The main theorem states that a right injective cogenerator ring R has strongly bounded basic ring R_0 , that is, every onesided ideal $\neq 0$ of R_0 contains an ideal $\neq 0$. A right injective cogenerator ring R is characterized by the condition:

- (1) R is semiperfect and right self-injective and
- (2) R has (finite) essential right socle.

We show that (2) can be replaced by

- (2') R_0 is strongly right bounded and has finite left socle.

Actually a right perfect right self-injective ring R is right PF iff R_0 is strongly bounded (Corollary 4).

We say that a module M has F(ES) if M has (finite) essential socle. Thus socle M (= sum of the minimal submodules) is essential when M has ES, and if M has FES, then socle M is a sum of finitely many simple modules. The Goldie dimension of a module M is the maximum number (= integer) n of nonzero independent submodules of M , when such exists. Then we write $\dim M = n$. (This is unambiguous by the Krull-Schmidt Theorem applied to the injective hull of M .) We introduce Goldie dimension in order to formalize several statements below. Note, however, that ES plus Goldie dimension $n < \infty \Rightarrow$ (F)ES

THEOREM 1. For a semiperfect right self-injective ring R with basic ring R_0 the f.a.e.:

- (1) (a) R is right PF (= R is an [injective] cogenerator in mod- R).
- (1) (b) R has right ES.
- (2) R_0 is strongly bounded with FES socle $S = V_1 \oplus \cdots \oplus V_n$, where V_i is an ideal which is a simple right, and a simple left, R_0 -module, $i = 1, \dots, n < \infty$.
- (3) R_0 is strongly bounded with left ES.
- (4) R_0 is strongly right bounded and every simple left R_0 -module V embeds in R_0 so that VR_0 has finite length as a left R_0 -module.
- (5) R_0 is strongly right bounded, and every simple left R_0 -module V of R_0 embeds in R_0 , and VR_0 contains a minimal ideal.
- (6) R_0 is strongly right bounded with left ES, and every minimal left ideal V generates an ideal R_0 of finite left Goldie dimension.
- (7) R_0 is strongly right bounded, and every simple left R -module V embeds in R

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so that VR has finite length as a left R -module (or contains a minimal ideal of R).

(8) R_0 is strongly right bounded. R has left ES, and every minimal left ideal V of R generates an ideal VR of finite left length.

(9) R_0 is strongly bounded, and R has left ES.

PROOF. Now by the equivalent conditions (PF_1) – (PF_5) stated in the Introduction of [1], we have $1(a) \Leftrightarrow 1(b)$.

(1) \Rightarrow (2). R_0 is strongly right bounded by [1, Theorem 1]. In order to prove that R has FES, it suffices to prove that R_0 has left FES inasmuch as $R_0 \sim R$ (similarity or Morita equivalence as defined in [1]), and the condition (F)ES is characterizable in $R\text{-mod}$ by the property that finitely generated projective modules have (F)ES. Hence assume $R = R_0$ (self-basic). Then $R/\text{rad } R$ is a product of fields, hence every maximal right ideal is an ideal. Since R is right self-injective, then every finitely generated left ideal V of R is a left annulet. Thus, V satisfies $V = {}^\perp(V^\perp)$, and V^\perp is contained in a maximal (right) ideal P . Since R is right PF, every right ideal is a right annulet, so that ${}^\perp P \neq 0$. Hence $V = {}^\perp(V^\perp)$ contains an ideal ${}^\perp P \neq 0$. This proves that R is strongly (left) bounded, and, moreover, that any simple left ideal V is an ideal. Since R is strongly right bounded by [1, Theorem 1], then V is a simple right ideal. Similarly, every simple right ideal is a simple left ideal. Thus the left socle $S_l = S_r$, the right socle. Since R has right FES by PF_2 , then $S = S_r = S_l = W_1 \oplus \cdots \oplus W_n$ for finitely many ideals W_i which are simple in $R\text{-mod}$ and in $\text{mod-}R$, $i = 1, \dots, n$. Let $L \neq 0$ be a left ideal. Since R is strongly left bounded then L contains an ideal $\neq 0$, so that $L \cap S \neq 0$, proving that R has left FES. This completes the proof of (1) \Rightarrow (2).

(2) \Rightarrow (3) is trivial.

(3) \Rightarrow (4). Let $R = R_0$. If V is any simple left ideal, then V is an ideal, hence $VR = V$ has length = 1. As in [1], we write $R = \bigoplus_{i=1}^n e_i R = R_0$ for right prindecs $e_i R$, $i = 1, \dots, n$. Let B_i be an ideal $\neq 0$ contained in $e_i R$, and using the fact that R has left ES, let W_i be a simple left ideal contained in B_i , $i = 1, \dots, n$. Since W_i is an ideal, and a minimal left ideal, and since R is right bounded, then W_i is a minimal right ideal. Moreover, since R is right self-injective, an isomorphism $W_i \approx W_j \Rightarrow \exists a \in R$ with $aW_i = W_j$ so $W_i \approx W_j$ for any $i \neq j$. Furthermore, $e_i R$ is an indecomposable injective, hence uniform, so that W_i is essential in $e_i R$, and hence $W_1 \oplus \cdots \oplus W_n$ is essential in R . This proves that the socle of R_R has length = n (hence by injectivity and self-basiness of R every simple right module $\hookrightarrow R$) so R is right PF by PF_3 (or PF_5). Then, every simple left module $V \hookrightarrow R$ by [1, Corollary 11]. This completes the proof of (3) \Rightarrow (4).

(4) \Rightarrow (5) is trivial.

(5) \Rightarrow (1). We may assume $R = R_0 = \bigoplus_{i=1}^n e_i R$ as above. Also let V_1, \dots, V_n be the simple left modules of R assumed to be embedded in R , and let B_i be a minimal ideal $\neq 0$ contained in $V_i R$, $i = 1, \dots, n$. Since R is strongly right bounded, B_i is actually a minimal right ideal, and by right self-injectivity,

we have $B_i \cong B_j$ for any $i \neq j, i = 1, \dots, n$. Thus, every simple right module $W \hookrightarrow R$, so that R is (PF₅).

(6) \Rightarrow (1). Since $R_0 = R$ is strongly right bounded, then every $e_i R$ contains an ideal $B_i \neq 0$. Since R has left ES, B_i contains an ideal $W_i = V_i R$, where V_i is a minimal left ideal $\neq 0, i = 1, \dots, n$. By the fact that $V_i R$ has finite left Goldie dimension, $W_i = V_i R$ is a finite direct sum of simple left ideals, and hence $V_i R$ has d.c.c. on left R -submodules, $i = 1, \dots, n$. Then, there is a minimal ideal B_i of R contained in W_i , and strong right boundedness of R implies that B_i is a minimal right ideal, $i = 1, \dots, n$. Moreover, $B_i \cong B_j, i = 1, \dots, n$, so every simple right R -module embeds in $R = R_0$. Since $e_i R$ is injective, B_i is essential in $e_i R, i = 1, \dots, n$, so R has right (F)ES. This completes the proof of (6) \Rightarrow (1).

(1) \Rightarrow (6) via (1) \Rightarrow (4) \Rightarrow (6).

(4) \Leftrightarrow (7) (or (5) \Leftrightarrow (7)) follows from the similarity $R_0 \sim R$, which results from the category equivalence $T: R_0\text{-mod} \approx R\text{-mod}$ where $T \approx Re_0 \otimes_{R_0}$ and $T^{-1} \approx e_0 R \otimes_R$. To see this note that any left R_0 -module V_0 is mapped onto $V = Re_0 \otimes_{R_0} V_0$, and that V is simple iff V_0 is simple. Moreover, every simple left R -module V comes from a simple $V_0 = T^{-1}V$. Furthermore, $V_0 \hookrightarrow R$ implies by exactness of T ($=$ flatness of Re_0 in $\text{mod-}R_0$) that $V \approx Re_0 V_0 = R V_0 \hookrightarrow R$. Thus, $V \hookrightarrow R$ for all simple $V \in R\text{-mod}$ iff $V_0 \hookrightarrow R$ for all simple $V_0 \in R_0\text{-mod}$.

To continue with (4) \Rightarrow (7), we have that (4) \Rightarrow (2) which by the remark made in the proof of (1) \Rightarrow (2) implies every finitely generated projective left R -module has FES. Thus, $RV =$ a direct sum of simple left ideals $\approx V$.has finite length.

To complete the equivalence of (4) \Leftrightarrow (7), we note that (5) \Leftrightarrow parenthetic part of (7) inasmuch as the Morita theorems imply that the correspondence

$$\text{ideals } R_0 \rightarrow \text{ideals } R,$$

$$I_0 \mapsto I = R I_0 R,$$

$$e_0 I e_0 \hookleftarrow I,$$

is a bijection. Thus, since $V_0 R_0$ maps into VR under this correspondence, we have the stated equivalence.

Now (7) \Rightarrow (4) can be proved since (7) $\Rightarrow VR$ contains a minimal ideal B which implies that $V_0 R_0$ contains a minimal ideal B_0 , so (7) \Rightarrow (5), and (5) \Rightarrow (4).

(6) \Leftrightarrow (8) inasmuch as R_0 has left ES iff R has left ES. (This remark also suffices for (3) \Leftrightarrow (9).) Then the equivalence of (6) and (8) follows as the equivalence of (4) and (7). (Note VR has finite length iff it has finite Goldie dimension.)

2. COROLLARY. *A semiperfect right FPF ring R with nil radical is right PF iff any of the equivalent conditions hold:*

2(a). *R has finite essential left socle.*

2(b). Every simple left R -module V embeds in R , and VR has finite length as a left R -module (or else VR contains a minimal ideal).

PROOF. R is right self-injective and R_0 is strongly right bounded by [1, Theorem 1]. Then (2) and (3) of Theorem 1 apply for (a), and (b) is equivalent to (7).

3. COROLLARY. A right perfect right FPF ring R is right PF iff R satisfies any of the following equivalent conditions:

- (a) R has finite left Goldie dimension.
- (b) Every simple left ideal generates an ideal of finite left length.
- (c) R has finite left socle.
- (d) Every ideal of R contains a minimal ideal.

PROOF. A right perfect ring R has left ES and nil radical so 2(a) \Leftrightarrow 3(a) and (c) \Leftrightarrow (a). Also 2(b) \Leftrightarrow 3(b). Clearly 3(d) \Rightarrow 2(b) (the parenthetical part), and conversely if every VR contains a minimal ideal, then so does every ideal $I \neq 0$, since I contains a minimal left ideal V .

4. COROLLARY. A right self-injective right perfect ring R is right PF iff the basic ring R_0 is (right and left) strongly bounded.

PROOF. R right perfect $\Rightarrow R_0$ right perfect $\Rightarrow R_0$ has left ES so the corollary is statement (3) of the theorem.

For emphasis, we deduce a theorem of Osofsky and Utumi (for R_R injective) and Tachikawa (for R_R FPF):

5. COROLLARY. Any right self-injective or right FPF left perfect ring R is right PF (and R_0 is strongly bounded).

PROOF. Right FPF left perfect is right self-injective. In this case, R has right ES, so R is right PF (by 1(b)), and R_0 is strongly bounded by Theorem 1(2).

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