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INJECTIVE COGENERATOR RINGS AND A THEOREM OF TACHIKAWA. II

CARL FAITH

ABSTRACT. The main theorem states that a right injective cogenerator ring R has strongly bounded basic ring R_0, that is, every onesided ideal $\neq 0$ of R_0 contains an ideal $\neq 0$. A right injective cogenerator ring R is characterized by the condition:

(1) R is semiperfect and right self-injective and

(2) R has (finite) essential right socle.

We show that (2) can be replaced by

(2′) R_0 is strongly right bounded and has finite left socle.

Actually a right perfect right self-injective ring R is right PF iff R_0 is strongly bounded (Corollary 4).

We say that a module M *has* F(ES) if M has (finite) essential socle. Thus socle M (= sum of the minimal submodules) is essential when M has ES, and if M has FES, then socle M is a sum of finitely many simple modules. The Goldie dimension of a module M is the maximum number (= integer) n of nonzero independent submodules of M, when such exists. Then we write $\dim M = n$. (This is unambiguous by the Krull-Schmidt Theorem applied to the injective hull of M.) We introduce Goldie dimension in order to formalize several statements below. Note, however, that ES plus Goldie dimension $n < \infty \Rightarrow$ (F)ES

THEOREM 1. *For a semiperfect right self-injective ring R with basic ring R_0 the f.a.e.*:

(1) (a) *R is right PF (= R is an [injective] cogenerator in* mod-R).

(1) (b) *R has right ES.*

(2) *R_0 is strongly bounded with FES socle $S = V_1 \oplus \cdots \oplus V_n$, where V_i is an ideal which is a simple right, and a simple left, R_0-module, $i = 1, \ldots, n < \infty$.*

(3) *R_0 is strongly bounded with left ES.*

(4) *R_0 is strongly right bounded and every simple left R_0-module V embeds in R_0 so that VR_0 has finite length as a left R_0-module.*

(5) *R_0 is strongly right bounded, and every simple left R_0-module V of R_0 embeds in R_0, and VR_0 contains a minimal ideal.*

(6) *R_0 is strongly right bounded with left ES, and every minimal left ideal V generates an ideal R_0 of finite left Goldie dimension.*

(7) *R_0 is strongly right bounded, and every simple left R-module V embeds in R*

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so that VR has finite length as a left R-module (or contains a minimal ideal of R).

(8) *R_0 is strongly right bounded. R has left ES, and every minimal left ideal V of R generates an ideal VR of finite left length.*

(9) *R_0 is strongly bounded, and R has left ES.*

PROOF. Now by the equivalent conditions (PF_1)–(PF_5) stated in the Introduction of [1], we have 1(a) $\Leftrightarrow$ 1(b).

(1) $\Rightarrow$ (2). R_0 is strongly right bounded by [1, Theorem 1]. In order to prove that R has FES, it suffices to prove that R_0 has left FES inasmuch as $R_0 \sim R$ (similarity or Morita equivalence as defined in [1]), and the condition (F)ES is characterizable in R-mod by the property that finitely generated projective modules have (F)ES. *Hence assume $R = R_0$ (self-basic).* Then $R/\operatorname{rad} R$ is a product of fields, hence every maximal right ideal is an ideal. Since R is right self-injective, then every finitely generated left ideal V of R is a left annulet. Thus, V satisfies $V = {}^{\perp}(V^{\perp})$, and $V^{\perp}$ is contained in a maximal (right) ideal P. Since R is right PF, every right ideal is a right annulet, so that ${}^{\perp}P \neq 0$. Hence $V = {}^{\perp}(V^{\perp})$ contains an ideal ${}^{\perp}P \neq 0$. This proves that R is strongly (left) bounded, and, moreover, that any simple left ideal V is an ideal. Since R is strongly right bounded by [1, Theorem 1], then V is a simple right ideal. Similarly, every simple right ideal is a simple left ideal. Thus the left socle $S_l = S_r$, the right socle. Since R has right FES by PF_2, then $S = S_r = S_l = W_1 \oplus \cdots \oplus W_n$ for finitely many ideals W_i which are simple in R-mod and in mod-R, $i = 1, \ldots, n$. Let $L \neq 0$ be a left ideal. Since R is strongly left bounded then L contains an ideal $\neq 0$, so that $L \cap S \neq 0$, proving that R has left FES. This completes the proof of (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4). Let $R = R_0$. If V is any simple left ideal, then V is an ideal, hence $VR = V$ has length $= 1$. As in [1], we write $R = \oplus_{i=1}^{n} e_i R = R_0$ for right prindecs $e_i R$, $i = 1, \ldots, n$. Let B_i be an ideal $\neq 0$ contained in $e_i R$, and using the fact that R has left ES, let W_i be a simple left ideal contained in B_i, $i = 1, \ldots, n$. Since W_i is an ideal, and a minimal left ideal, and since R is right bounded, then W_i is a minimal right ideal. Moreover, since R is right self-injective, an isomorphism $W_i \approx W_j \Rightarrow \exists a \in R$ with $aW_i = W_j$ so $W_i \not\approx W_j$ for any $i \neq j$. Furthermore, $e_i R$ is an indecomposable injective, hence uniform, so that W_i is essential in $e_i R$, and hence $W_1 \oplus \cdots \oplus W_n$ is essential in R. This proves that the socle of R_R has length $= n$ (hence by injectivity and self-basicness of R every simple right module $\hookrightarrow R$) so R is right PF by PF_3 (or PF_5). Then, every simple left module $V \hookrightarrow R$ by [1, Corollary 11]. This completes the proof of (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (5) is trivial.

(5) $\Rightarrow$ (1). We may assume $R = R_0 = \oplus_{i=1}^{n} e_i R$ as above. Also let $V_1, \ldots, V_n$ be the simple left modules of R assumed to be embedded in R, and let B_i be a minimal ideal $\neq 0$ contained in $V_i R$, $i = 1, \ldots, n$. Since R is strongly right bounded, B_i is actually a minimal right ideal, and by right self-injectivity,

we have $B_i \not\approx B_j$ for any $i \neq j$, $i = 1, \dots, n$. Thus, every simple right module $W \hookrightarrow R$, so that R is (PF_5).

(6) $\Rightarrow$ (1). Since $R_0 = R$ is strongly right bounded, then every $e_i R$ contains an ideal $B_i \neq 0$. Since R has left ES, B_i contains an ideal $W_i = V_i R$, where V_i is a minimal left ideal $\neq 0$, $i = 1, \dots, n$. By the fact that $V_i R$ has finite left Goldie dimension, $W_i = V_i R$ is a finite direct sum of simple left ideals, and hence $V_i R$ has d.c.c. on left R-submodules, $i = 1, \dots, n$. Then, there is a minimal ideal B_i of R contained in W_i, and strong right boundedness of R implies that B_i is a minimal right ideal, $i = 1, \dots, n$. Moreover, $B_i \not\approx B_j$, $i = 1, \dots, n$, so every simple right R-module embeds in $R = R_0$. Since $e_i R$ is injective, B_i is essential in $e_i R$, $i = 1, \dots, n$, so R has right (F)ES. This completes the proof of (6) $\Rightarrow$ (1).

(1) $\Rightarrow$ (6) via (1) $\Rightarrow$ (4) $\Rightarrow$ (6).

(4) $\Leftrightarrow$ (7) (or (5) $\Leftrightarrow$ (7)) follows from the similarity $R_0 \sim R$, which results from the category equivalence T: R_0-mod $\approx$ R-mod where $T \approx Re_0 \otimes_{R_0}$ and $T^{-1} \approx e_0 R \otimes_R$. To see this note that any left R_0-module V_0 is mapped onto $V = Re_0 \otimes_{R_0} V_0$, and that V is simple iff V_0 is simple. Moreover, every simple left R-module V comes from a simple $V_0 = T^{-1}V$. Furthermore, $V_0 \hookrightarrow R$ implies by exactness of T (= flatness of Re_0 in mod-R_0) that $V \approx Re_0 V_0 = RV_0 \hookrightarrow R$. Thus, $V \hookrightarrow R$ for all simple $V \in R$-mod iff $V_0 \hookrightarrow R$ for all simple $V_0 \in R_0$-mod .

To continue with (4) $\Rightarrow$ (7), we have that (4) $\Rightarrow$ (2) which by the remark made in the proof of (1) $\Rightarrow$ (2) implies every finitely generated projective left R-module has FES. Thus, RV = a direct sum of simple left ideals $\approx V$.has finite length.

To complete the equivalence of (4) $\Leftrightarrow$ (7), we note that (5) $\Leftrightarrow$ parenthetic part of (7) inasmuch as the Morita theorems imply that the correspondence

$$\text{ideals } R_0 \to \text{ideals } R,$$

$$I_0 \mapsto I = RI_0R,$$

$$e_0 I e_0 \leftarrow\!\shortmid I,$$

is a bijection. Thus, since $V_0 R_0$ maps into VR under this correspondence, we have the stated equivalence.

Now (7) $\Rightarrow$ (4) can be proved since (7) $\Rightarrow$ VR contains a minimal ideal B which implies that $V_0 R_0$ contains a minimal ideal B_0, so (7) $\Rightarrow$ (5), and (5) $\Rightarrow$ (4).

(6) $\Leftrightarrow$ (8) inasmuch as R_0 has left ES iff R has left ES. (This remark also suffices for (3) $\Leftrightarrow$ (9).) Then the equivalence of (6) and (8) follows as the equivalence of (4) and (7). (Note VR has finite length iff it has finite Goldie dimension.)

2. COROLLARY. *A semiperfect right FPF ring R with nil radical is right PF iff any of the equivalent conditions hold*:

2(a). *R has finite essential left socle.*

2(b). *Every simple left R-module V embeds in R, and VR has finite length as a left R-module (or else VR contains a minimal ideal).*

PROOF. R is right self-injective and R_0 is strongly right bounded by [1, Theorem 1]. Then (2) and (3) of Theorem 1 apply for (a), and (b) is equivalent to (7).

3. COROLLARY. *A right perfect right FPF ring R is right PF iff R satisfies any of the following equivalent conditions:*

(a) *R has finite left Goldie dimension.*

(b) *Every simple left ideal generates an ideal of finite left length.*

(c) *R has finite left socle.*

(d) *Every ideal of R contains a minimal ideal.*

PROOF. A right perfect ring R has left ES and nil radical so 2(a) $\Leftrightarrow$ 3(a) and (c) $\Leftrightarrow$ (a). Also 2(b) $\Leftrightarrow$ 3(b). Clearly 3(d) $\Rightarrow$ 2(b) (the parenthetical part), and conversely if every VR contains a minimal ideal, then so does every ideal $I \neq 0$, since I contains a minimal left ideal V.

4. COROLLARY. *A right self-injective right perfect ring R is right PF iff the basic ring R_0 is (right and left) strongly bounded.*

PROOF. R right perfect $\Rightarrow$ R_0 right perfect $\Rightarrow$ R_0 has left ES so the corollary is statement (3) of the theorem.

For emphasis, we deduce a theorem of Osofsky and Utumi (for R_R injective) and Tachikawa (for R_R FPF):

5. COROLLARY. *Any right self-injective or right FPF left perfect ring R is right PF (and R_0 is strongly bounded).*

PROOF. Right FPF left perfect is right self-injective. In this case, R has right ES, so R is right PF (by 1(b)), and R_0 is strongly bounded by Theorem 1(2).

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903