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SETS WHICH CAN BE EXTENDED TO m-CONVEX SETS

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ABSTRACT. Let S be a compact set in R^d, $T_0 \subseteq S$. Then T_0 lies in an m-convex subset of S if and only if every finite subset of T_0 lies in an m-convex subset of S. For S a closed set in R^d and $T_0 \subseteq S$, let $T_1 = \{P\colon P$ a polytope in S having vertex set in T_0, $\dim P \leqslant d - 1\}$. If for every three members of T_1, at least one of the corresponding convex hulls

$$\operatorname{conv}\{P_i \cup P_j\}, \qquad 1 \leqslant i < j \leqslant 3.$$

lies in S, then T_0 lies in a 3-convex subset of S. An analogous result holds for m-convex sets provided $\ker S \neq \emptyset$.

1. **Introduction.** Let S be a subset of some linear topological space. The set S is said to be *m-convex*, $m \geqslant 2$, if and only if for every m-member subset of S, at least one of the $\binom{m}{2}$ line segments determined by these points lies in S. A point x in S is called a *point of local convexity* of S if and only if there is some neighborhood N of x such that if $y, z \in N \cap S$, then $[y, z] \subseteq S$. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) of S.

Several interesting decomposition theorems have been obtained for closed m-convex sets in the plane (Valentine [5], Stamey and Marr [3], Breen and Kay [1]). However, for S a set in R^d and $T \subseteq S$, little work has been done to determine sufficient conditions under which T may be extended to an m-convex subset of S. Characterizations of this kind allow us to determine when T is contained in a starshaped subset of S for S compact [4] and similarly to determine when T lies in a union of k convex subsets of S (a variation of [2, Theorem 2]). Hence the purpose of this paper is to obtain an analogue of these results for m-convex sets.

The following familiar terminology will be used: Throughout the paper, $\operatorname{conv} S$, $\operatorname{cl} S$, and $\ker S$ will denote the convex hull, closure, and kernel, respectively, of the set S. Also, if S is convex, $\dim S$ will denote the dimension of S, and dist will denote the Euclidean metric for R^d.

2. **A characterization theorem.** Our first result will require the following easy lemma.

LEMMA 1. *If A is an m-convex set in R^d, then $(A)_p \equiv \{x\colon \operatorname{dist}(A, x) < p\}$ is also m-convex for every $p > 0$.*

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PROOF. Let $x_1, \ldots, x_m$ belong to $(A)_p$ for some $p > 0$. Then for each x_i, there is some y_i in A such that $\text{dist}(x_i, y_i) < p$. Since A is m-convex, one of the segments determined by the y points is in A, say $[y_1, y_2] \subseteq A$. But then for each λ, $0 \leqslant \lambda \leqslant 1$, $\lambda x_1 + (1 - \lambda)x_2$ lies in a p-neighborhood of $\lambda y_1 + (1 - \lambda)y_2$, so $\lambda x_1 + (1 - \lambda)x_2 \in (A)_p$, and $[x_1, x_2] \subseteq (A)_p$.

THEOREM 1. *Let S be a compact subset of R^d, $T \subseteq S$. Then T lies in an m-convex subset of S if and only if every finite subset of T lies in an m-convex subset of S.*

PROOF. The necessity is obvious. The sufficiency will require the Hausdorff metric defined on the collection of compact subsets of S. Let $\mathcal{F} = \{A: A$ compact and $A \subseteq S\}$, and let d denote the Hausdorff metric for $\mathcal{F}$. That is, if $(A)_p = \{x: x \in S$ and $\text{dist}(x, A) < p\}$, then for A, B in $\mathcal{F}$,

$$d(A, B) = \inf\{p: A \subseteq (B)_p \text{ and } B \subseteq (A)_p, p > 0\}.$$

Since S is bounded, it is easy to show that $\mathcal{F}$ is bounded with respect to the Hausdorff metric, and standard arguments similar to those given in Valentine [4, pp. 37–39] may be used to show that $(\mathcal{F}, d)$ has the Bolzano-Weierstrass property. That is, every infinite subset of $\mathcal{F}$ has a limit point F in $\mathcal{F}$, and $F \neq \varnothing$.

Now for each x in T, define $\mathcal{C}_x = \{A: A$ compact and m-convex, with $x \in A \subseteq S\}$. It is easy to see that the collection of $\mathcal{C}_x$ sets has the finite intersection property: For $\mathcal{C}_1, \ldots, \mathcal{C}_k$ any finite collection of these sets, with $x_1, \ldots, x_k$ the corresponding members of T, by hypothesis there is some m-convex subset B of S containing $x_1, \ldots, x_k$. Standard arguments show that cl B is m-convex, and since S is compact, cl B is a compact subset of S. Hence cl $B \in \mathcal{C}_i$, $1 \leqslant i \leqslant k$.

Furthermore, we assert that each $\mathcal{C}_x$ set is compact, and clearly it suffices to show that every infinite subset $\{A_n\}$ in $\mathcal{C}_x$ has a limit point in $\mathcal{C}_x$. Since $\mathcal{C}_x \subseteq \mathcal{F}$, by our opening paragraph $\{A_n\}$ has a limit point $A \neq \varnothing$ in $\mathcal{F}$, so it remains to show that A is m-convex and $x \in A$.

Without loss of generality, assume that $\lim A_n = A$. That is,

$$\lim d(A_n, A) = 0.$$

Select a sequence $\{p_n\}$ of positive numbers converging to zero so that $d(A_n, A) < p_n$ for each n. Then $A \subseteq (A_n)_{p_n}$ and $A_n \subseteq (A)_{p_n}$. Select points $x_1, \ldots, x_m$ in A. By the Lemma, since A_n is m-convex, $(A_n)_{p_n}$ is m-convex, so for each n, there exists some pair $y(n)$, $z(n)$ in $\{x_1, \ldots, x_m\}$ such that $[y(n), z(n)] \subseteq (A_n)_{p_n}$. Hence one of the $\binom{m}{2}$ pairs from $\{x_1, \ldots, x_m\}$ appears infinitely many times, and we may choose a subsequence I of natural numbers and a pair y_0, z_0 in $\{x_1, \ldots, x_m\}$ so that $[y_0, z_0] \subseteq (A_n)_{p_n}$ for all $n \in I$. Then

$$[y_0, z_0] \subseteq (A_n)_{p_n} \subseteq ((A)_{p_n})_{p_n} = (A)_{2p_n} \quad \text{for } n \in I.$$

But $\lim(p_n\colon n \in I) = 0$, so $[y_0, z_0] \subseteq \bigcap\{(A)_{2p_n}\colon n \in I\} = A$. Thus A is m-convex. A similar argument reveals that $x \in A$, so $A \in \mathcal{C}_x$.

Therefore every infinite subset of $\mathcal{C}_x$ converges to a member of $\mathcal{C}_x$, and $\mathcal{C}_x$ is compact. Hence $\{\mathcal{C}_x\colon x \in T\}$ is a collection of compact sets having the finite intersection property, and $\bigcap\{\mathcal{C}_x\colon x \in T\} \neq \varnothing$. For A in this intersection, A is an m-convex subset of S which contains T, finishing the proof of the theorem.

REMARK. It is interesting to notice that Theorem 1 is valid when R^d is replaced by an arbitrary Banach space.

3. **The 3-convex case.** The following theorem gives sufficient conditions for a subset T_0 of S to lie in a 3-convex subset of S.

THEOREM 2. *Let S be a closed subset of R^d, $T_0 \subseteq S$, and let $T_1 = \{P\colon P$ a polytope in S having vertex set in T_0, $\dim P \leqslant d - 1\}$. If for every three members of T_1, at least one of the corresponding convex hulls* $\operatorname{conv}\{P_i \cup P_j\}$, $1 \leqslant i < j \leqslant 3$, *lies in S, then T_0 lies in a 3-convex subset of S.*

PROOF. Define U to be the union of T_1 together with all the simply connected regions bounded by $(d-1)$-polytopes contained in members of T_1. Let $T = \operatorname{cl} U$. We assert that T is the required 3-convex subset of S.

To see that $U \subseteq S$, note that for every point x in U, either x is in T_1 (and hence in S) or x is interior to some region R which is bounded by a subset of $\bigcup T_1$. In the latter case, R has as its boundary a union of $(d-1)$-polytopes, each contained in a member of T_1, and an easy inductive argument shows that $R \subseteq S$. Then $U \subseteq S$, and since S is closed, $T = \operatorname{cl} U \subseteq S$.

Thus it remains to show only that T is 3-convex. Now if T is not connected, then clearly our hypothesis will imply that T consists of two components, each convex, and T will indeed be 3-convex. Hence we assume that T is connected and nonconvex.

Since T is closed, connected, and nonconvex, T cannot be locally convex [4], so T contains at least one point of local nonconvexity. We assert that for q an lnc point of T, $q \in \ker T$: For any neighborhood N of q, there are points t_1, t_2 in U with $[t_1, t_2] \not\subseteq T$. Now t_1, t_2 are in simply connected subsets of S bounded by polytopes which are contained in members of T_1, so $[t_1, t_2]$ intersects polytopes P_1, P_2 of T_1 for which $\operatorname{conv}\{P_1 \cup P_2\} \not\subseteq S$. Since such a pair P_{1n}, P_{2n} may be selected for every neighborhood $N(q, 1/n)$ of q having radius $1/n$, in this manner we select sequences $\{P_{1n}\}$, $\{P_{2n}\}$ such that each P_{in} is a member of T_1 nondisjoint from $N(q, 1/n)$, and so that

$$\operatorname{conv}\{P_{1n} \cup P_{2n}\} \not\subseteq S \quad \text{for each } n.$$

For z in T, $z \neq q$, there is a sequence $\{u_n\}$ in U converging to z, and the ray $R(q, u_n)$ emanating from q through u_n intersects some member P_{3n} of T_1 at a point u_n', where $q < u_n \leqslant u_n'$. Then by hypothesis, for at least one of $i = 1$ or $i = 2$, $\operatorname{conv}\{P_{1n} \cup P_{3n}\} \subseteq S$. Hence for one of $i = 1$ or $i = 2$, say for

$i = 1$, $\operatorname{conv}\{P_{1n} \cup P_{3n}\} \subseteq S$ for all n in some infinite index set I. And so $\operatorname{conv}\{P_{1n} \cup P_{3n}\} \subseteq T$ for these n. Since every neighborhood of q contains points in all but a finite number of the P_{1n} sets, this implies that

$$[q, u_n] \subseteq \operatorname{conv}\{q \cup P_{3n}\} \subseteq \operatorname{cl} \operatorname{conv}\{P_{1n} \cup P_{3n}\} \subseteq \operatorname{cl} T = T$$

whenever $n \in I$. Thus for $n \in I$, $[q, u_n] \subseteq T$, and $[q, z] \subseteq T$. Hence $q \in \ker T$ and our assertion is proved.

Finally, to complete the proof that T is 3-convex, let x_1, x_2, x_3 be in T and again let q be an lnc point of T. Now if any x_i is q, the argument is trivial, so assume that $x_i \neq q$, $i = 1, 2, 3$. There are sequences $\{y_{1n}\}, \{y_{2n}\}, \{y_{3n}\}$ in $U \sim \{q\}$ converging to x_1, x_2, x_3, respectively, and the ray $R(q, y_{in})$ cuts some member P_{in} of T_1 at y'_{in}, $q < y_{in} \leqslant y'_{in}$. For each n, there correspond an appropriate i and j, $1 \leqslant i < j \leqslant 3$, for which $\operatorname{conv}\{P_{in} \cup P_{jn}\} \subseteq U$. Hence for some infinite subset I of natural numbers and for at least one pair i, j in $\{1, 2, 3\}$, say for $i = 1$ and $j = 2$, $\operatorname{conv}\{P_{1n} \cup P_{2n}\} \subseteq U$ whenever $n \in I$. Since $q \in \ker T$, it follows that $\operatorname{conv}\{q \cup P_{1n} \cup P_{2n}\} \subseteq T$ for all $n \in I$. Then $\operatorname{conv}\{q, y_{1n}, y_{2n}\} \subseteq T$ for $n \in I$, and $\operatorname{conv}\{q, x_1, x_2\} \subseteq T$. Hence $[x_1, x_2] \subseteq T$ and T is indeed a 3-convex subset of S, finishing the proof of Theorem 2.

The following corollary is a direct application of Theorem 2 to the planar case.

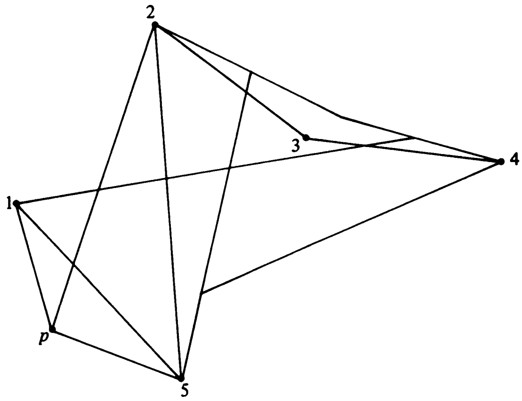
COROLLARY. *Let S be a closed subset of the plane, $T_0 \subseteq S$, and let $T_1 = \{[x, y]: x, y$ in T_0 and $[x, y] \subseteq S\}$. Assume that for every three segments s_1, s_2, s_3 (possibly degenerate) in T_1, at least one of the corresponding convex hulls $\operatorname{conv}\{s_i \cup s_j\}$, $1 \leqslant i < j \leqslant 3$, lies in S. Then T_0 lies in a 3-convex subset of S.*

Note that the condition stated in Theorem 2 and its corollary is sufficient but not necessary for T_0 to lie in a 3-convex subset of S, as Example 1 illustrates.

EXAMPLE 1. Let S be the boundary of a nondegenerate triangle A in the plane, T_0 the vertex set of A. Then clearly T_1 is the set of edges of A and cannot satisfy the hypothesis of Theorem 2, yet T_0 lies in a union of two convex subsets of S. (Consider a vertex and its opposite edge.)

Also, it is interesting to notice that the dimension of the sets in T_1 above cannot be reduced. In the planar case, for every three points of T_0, one of the corresponding segments may lie in S without T_0 being contained in a 3-convex subset of S. In fact we have a more surprising result: Even if every three points of $\bigcup T_1$ have a corresponding segment in S, the result of Theorem 2 fails, as the following example reveals.

EXAMPLE 2. Let S be the set in Figure 1, $T_0 = \{1, 2, 3, 4, 5\}$, and let $T_1 = \{[x, y]: x, y \in T_0$ and $[x, y] \subseteq S\}$. Then for every three points in $\bigcup T_1$, one of the corresponding segments is in S, yet T_0 does not lie in a 3-convex subset of S. (Otherwise, since $p \in \ker S \cap \operatorname{bdry} S$ and q is not an lnc point, by [1, Theorem 1, Corollary 2], T_0 could be partitioned into sets A, B so that $\operatorname{conv} A \cup \operatorname{conv} B \subseteq S$, clearly impossible.)



In conclusion, the proof of Theorem 2 may be adapted appropriately to yield the following result for m-convex sets.

THEOREM 3. *Let S be a closed subset of R^d with* $\ker S \neq \emptyset$. *Let $T_0 \subseteq S$ and let $T_1 = \{P$: P a polytope in S having vertex set in T_0,* $\dim P \leqslant d - 1\}$. *If for every m members of T_1, at least one of the corresponding convex hulls*

$$\operatorname{conv}\{P_i \cup P_j\}, \qquad 1 \leqslant i < j \leqslant m,$$

lies in S, then T_0 lies in an m-convex subset of S.

PROOF. Define T as in the proof of Theorem 2 and let $T' = \bigcup\{[p, t]$: $t \in T\}$, where $p \in \ker S$. Using techniques similar to those in Theorem 2, it is easy to show that T' is the required m-convex subset of S.

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