FIBRATIONS OVER A CWh-BASE

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Abstract. This note provides a short argument for the known fact that the total space of a fibration has the homotopy type of a CW-complex if base and fiber have.

1. Notation. \( F \to E \to B \) is a (Hurewicz) fibration. A CWh-space is a space having the homotopy type of a CW-complex. The following result is due to Stasheff [9, Proposition (0)].

2. Theorem. \( E \) is a CWh-space if \( F \) and \( B \) are.

Proof. We replace the inductive construction of [9] by the CW approximation theorem [8, p. 412] that is due to Whitehead [11]: to the topological space \( E \) there exists a CW-complex \( X \), called a CW-substitute for \( E \) in [10, p. 97], and a weak homotopy equivalence \( f: X \to E \). We make \( f \) into a fibration by taking the associated mapping path fibration \( q: P_f \to E \), see e.g. [8, p. 99]. Then \( q \) is a weak homotopy equivalence too, and \( P_f \) is a CWh-space. Therefore \( pq \) is a fibration with a CWh-fiber by 3 below. Hence \( q \) induces a genuine homotopy equivalence between the fibers of \( pq \) and \( p \) and is therefore a fiber homotopy equivalence by [3, 6.3].

3. Proposition. \( F \) is a CWh-space if \( E \) and \( B \) are.

Proof. Compare [9, Corollary (13)]. By coglueing homotopy equivalences, see [4, (1.2)] or [5, (8.7)], the pullbacks of the horizontal rows in the following diagram are homotopy equivalent.

\[ \begin{array}{ccc}
* & \to & B \\
\downarrow & & \downarrow \\
P B & \to & B \\
\downarrow & & \downarrow \\
P Z_p & \to & Z_p \\
 & \to & E \\
\end{array} \]

\( fi \) is the standard factorization of \( p \) over its mapping cylinder \( Z_p \), \( PZ_p \to Z_p \), \( PB \to B \) are the fibrations of paths starting from a point \( b \in Z_p \), resp.

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1 As it is remarked in [7, p. 27] Stasheff's proof is not correct, but can be patched.
\( f(b) = \ast \in B \), and the other arrows are obvious. The upper pullback is the fiber \( F \) (over \( \ast \)), the lower one is the space of the paths on the CWh-space \( Z_p \) starting from \( b \) and ending in \( E \subset Z_p \), and is therefore a CWh-space by [6].

4. Remark. If we assume that \( F \) and \( E \) are CWh-spaces, then the following is true: (a) \( B \) is not a CWh-space in general. Fiber and total spaces of Example 2.4.8 of [8, p. 77] are contractible, but the base space, the “Warsaw circle”, is not contractible, because it has the nonvanishing Čech homotopy group \( \tilde{\pi}_1(B) \cong \mathbb{Z} \) [2, §6]. (b) the loop space \( \Omega B \) is a CWh-space, because it is homotopy equivalent to the fiber of the inclusion \( F \to E \) [10, 2.56], and by delooping homotopy equivalences, see [1], \( B \) is a CWh-space too, if it is path-connected and has a numerable, null homotopic covering.

References