PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 62, Number 1, January 1977

FIBRATIONS OVER A CWh-BASE

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ABSTRACT. This note provides a short argument for the known fact that the total space of a fibration has the homotopy type of a CW-complex if base and fiber have.

1. NOTATION. $F \to E \to B$ is a (Hurewicz) fibration. A CWh-space is a space having the homotopy type of a CW-complex. The following result is due to Stasheff [9, Proposition (0)].[1]

2. THEOREM. *E is a CWh-space if F and B are.*

PROOF. We replace the inductive construction of [9] by the CW approximation theorem [8, p. 412] that is due to Whitehead [11]: to the topological space E there exists a CW-complex X, called ' CW-substitute for E' in [10, p. 97], and a weak homotopy equivalence f: $X \to E$. We make f into a fibration by taking the associated mapping path fibration q: $P_f \to E$, see e.g. [8, p. 99]. Then q is a weak homotopy equivalence too, and P_f is a CWh-space. Therefore pq is a fibration with a CWh-fiber by 3 below. Hence q induces a genuine homotopy equivalence between the fibers of pq and p and is therefore a fiber homotopy equivalence by [3, 6.3].

3. PROPOSITION. *F is a CWh-space if E and B are.*

PROOF. Compare [9, Corollary (13)]. By coglueing homotopy equivalences, see [4, (1.2)] or [5, (8.7)], the pullbacks of the horizontal rows in the following diagram are homotopy equivalent.

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & B \arrow[d] & E \arrow[l] \arrow[d] \\
PB \arrow[r] & B & E \arrow[l, "p"'] \\
PZ_p \arrow[r] \arrow[u] & Z_p \arrow[u, "f"] & E \arrow[l, "i"] \arrow[u]
\end{tikzcd}$$

fi is the standard factorization of p over its mapping cylinder Z_p, $PZ_p \to Z_p$, $PB \to B$ are the fibrations of paths starting from a point $b \in Z_p$, resp.

Received by the editors March 8, 1976 and, in revised form, July 19, 1976.
AMS (MOS) subject classifications (1970). Primary 55F05, 54E60.

[1] As it is remarked in [7, p. 27] Stasheff's proof is not correct, but can be patched.

$f(b) = * \in B$, and the other arrows are obvious. The upper pullback is the fiber F (over $*$), the lower one is the space of the paths on the CWh-space Z_p starting from b and ending in $E \subset Z_p$, and is therefore a CWh-space by [6].

4. REMARK. If we assume that F and E are CWh-spaces, then the following is true: (a) B is not a CWh-space in general. Fiber and total spaces of Example 2.4.8 of [8, p. 77] are contractible, but the base space, the "Warsaw circle", is not contractible, because it has the nonvanishing Čech homotopy group $\check{\pi}_1(B) \cong \mathbf{Z}$ [2, §6]. (b) the loop space ΩB is a CWh-space, because it is homotopy equivalent to the fiber of the inclusion $F \to E$ [10, 2.56], and by delooping homotopy equivalences, see [1], B is a CWh-space too, if it is path-connected and has a numerable, null homotopic covering.

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