PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 62, Number 1, January 1977

APPLICATIONS OF GROUP ACTIONS ON FINITE COMPLEXES TO HILBERT CUBE MANIFOLDS

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ABSTRACT. For any compact Hilbert cube manifold M such that $\tilde{H}_*(M, Z_p) = 0$, there exists an embedding g of M into the Hilbert cube Q such that $g(M)$ is the fixed point set of a semifree periodic homeomorphism of Q with period p. A counterexample is given to the conjecture that any two proper homotopic period p homeomorphisms of a Hilbert cube manifold such that the homeomorphisms revolve trivially about a unique fixed point are equivalent. A counterexample is also given for the case where the fixed point set is empty.

I. **Introduction.** This paper continues the study of group actions on Hilbert cube (Q) manifolds which essentially began with the work of R. Y. T. Wong [12]. That this study has begun in earnest is shown by the list of questions in [1]. Another recent paper in this area is [7].

The first part of this paper shows that certain finite-dimensional results generalize to Q manifolds without the obstruction theory necessary in the finite-dimensional case.

The second part of this paper demonstrates that Wong's classification [12] of certain periodic homeomorphisms of Q with unique fixed point does not generalize to Q manifolds. Thus the theory of such actions on Q manifolds should be richer than the theory of such actions on Q.

II. **Definitions and notation.** $Q = \prod_1^\infty[-1, 1]$, the Hilbert cube. A Hilbert cube manifold is a paracompact Hausdorff space admitting a cover of open sets homeomorphic to open sets of Q.

$f \simeq g$ means f is homotopic to g. f is properly homotopic to g if there is a proper homotopy between f and g. (A map is proper if the inverse image of every compact set is compact.)

$X \cong Y$ means X is homeomorphic to Y. Simple homotopy theory will be used. See [6] for a general reference. $X \simeq_S Y$ means X and Y have the same simple homotopy type.

proj $\lim\{X_i, C_i\}$ = the inverse limit of the spaces X_i with bonding maps C_i.

In the following definitions, let H be a homeomorphism of a space X with

Presented to the Society, June 11, 1975 under the title *Applications of group actions on finite complexes to Q manifolds*; received by the editors December 15, 1975.

AMS (MOS) subject classifications (1970). Primary 57E25, 57A20; Secondary 58B05, 54H15, 54H25.

Key words and phrases. Hilbert cube manifolds, semifree periodic homeomorphisms, fixed point set, simple homotopy equivalence, CW-complex.

period p; that is, p is the smallest integer such that $H^p = \mathrm{id}$.

The fixed point set of H is $\{x: h(x) = x\}$.

H is semifree if for all x such that

$$H(x) \neq x, \quad x, \quad H(x), H^2(x), \ldots, H^{p-1}(x)$$

are p distinct points.

H revolves trivially about its fixed point set K if for every neighborhood U of K, there is a contractible neighborhood $V \subset U$ of K such that $H(V) = V$.

$X/\{H\}$ = the quotient of X by the group generated by H, i.e., $X/\sim$ where $x \sim H^i(x)$, $1 \leqslant i \leqslant p-1$, for all $x \in X$.

$X/\{H(x_0)\} = X/\sim$ where $x_0 \sim H^i(x_0)$, $1 \leqslant i \leqslant p-1$.

Two period p homeomorphisms f_1 and f_2 of the space are equivalent if there exists a homeomorphism g such that $f_2 = gf_1g^{-1}$.

III. Q manifolds as fixed point sets.

THEOREM 1. *Let M be a compact Q manifold satisfying $\tilde{H}_*(M, Z_p) = 0$ (the reduced singular homology with coefficients in the integers modulo p). There is an embedding g: $M \to Q$ and a semifree, periodic homeomorphism H: $Q \to Q$ of period p such that M is exactly the fixed point set of H.*

PROOF. Lowell Jones [9] proved that if K is a finite CW-complex satisfying $H_*(K, Z_p) = 0$, then there exists a finite contractible CW-complex X containing K as a subcomplex and a semifree, period p homeomorphism g: $X \to X$ such that K is the fixed point set of g.

Now T. A. Chapman's Triangulation Theorem for Q manifolds [5] states that if M is a compact Q manifold, then there exist K, a finite simplicial (hence CW-) complex, and a homeomorphism f: $M \to K \times Q$. So $\tilde{H}_*(M, Z_p) = \tilde{H}_*(K, Z_p)$.

For this K, let h and X be as in the conclusion to Jones' result. Let i: $K \to X$ be the inclusion map. Let $g = i \times \mathrm{id}$: $K \times Q \to X \times Q$. Let $H = h \times \mathrm{id}$: $X \times Q \to X \times Q$. By a theorem of J. E. West [11], $X \times Q \cong Q$. So g is the desired embedding and H is the desired semifree periodic homeomorphism of period p. $\square$

IV. **Counterexamples.** R. Y. T. Wong [12] proved that if f, g are semifree period p homeomorphisms on Q with a unique fixed point and revolve trivially about the fixed point, then f and g are equivalent. In [1], it was asked: If f and g are semifree period p homeomorphisms of a Q manifold M such that (1) they have the same fixed point set K and it is a single point or empty, (2) both f and g revolve trivially at K, and (3) f is properly homotopic to g, then is f equivalent to g?

The answer to both cases, $K = \varnothing$ and K a single point, is no. The counterexample for the case $K = \varnothing$ is a straightforward construction using lens spaces [6, Chapter V]. The counterexample for the second case will take a bit more work.

COUNTEREXAMPLE 1. Let $S^3 = \Sigma_1 * \Sigma_2$ = the join of two copies of

$$S^1 = \{(te^{ix}, (1-t)e^{iy}) \in R^2 \times R^2 \colon 0 \leqslant x, y \leqslant 2\pi, 0 \leqslant t \leqslant 1\}.$$

Let $f_1 \colon S^3 \to S^3$ be defined by

$$f_1(te^{ix}, (1-t)e^{iy}) = (te^{ix+2\pi/7}, (1-t)e^{iy+2\pi/7}).$$

Let $f_2 \colon S^3 \to S^3$ be defined by

$$f_2(te^{ix}, (1-t)e^{iy}) = (te^{ix+4\pi/7}, (1-t)e^{iy+2\pi/7}).$$

Let $M = S^3 \times Q$, be a Q manifold.

Clearly g can be defined such that $g \colon S^3 \times I \to S^3$, $g_0 = f_1$, $g_1 = f_2$. So $f_1 \simeq f_2$. Define $F_i \colon M \to M$ by $F_i = f_i \times \mathrm{id}_Q$, $i = 1, 2$. Each F_i is a semifree period 7 homeomorphism of M. Define $G \colon F_1 \simeq F_2$ by $G = g \times \mathrm{id}_Q$. So conditions (1) and (3) of the question are met and (2) is vacuously satisfied. (The spaces involved are compact so all homotopies are proper.)

Suppose there exists H such that $F_2 = HF_1H^{-1}$. Consider the following commutative diagram where h is the naturally defined homeomorphism induced by H and $q_i \colon M \to M/\{F_i\}$ is a naturally defined quotient map, $i = 1, 2$.

$$\begin{tikzcd}
M \arrow[rr, "q_1"] \arrow[dr, "H"] \arrow[dd, "F_1"'] & & M/\{F_1\} \arrow[dr, "h"] \arrow[dd, "\mathrm{id}"] & \\
& M \arrow[rr, "q_2"] \arrow[dd, "F_2"] & & M/\{F_2\} \arrow[dd, "\mathrm{id}"] \\
M \arrow[rr, "q_1"] \arrow[dr, "H"] & & M/\{F_1\} \arrow[dr, "h"] & \\
& M \arrow[rr, "q_2"] & & M/\{F_2\}
\end{tikzcd}$$

If H existed, $M/\{F_1\} \cong M/\{F_2\}$. But $M/\{F_i\} \cong S^3/\{f_i\} \times Q$, $i = 1, 2$. By [6, §27], $S^3/\{f_1\} \cong L(7, 1)$, $S^3/\{f_2\} \cong L(7, 2)$, where $L(p, q)$ is the standard 3-dimensional lens space.

Chapman's Classification Theorem [5] states that if X and y are compact, connected Q manifolds and if $X \cong K \times Q$, and $Y \cong L \times Q$ (K, L finite simplicial complexes) are any two triangulations, then $X \cong Y$ if and only if K and L have the same simple homotopy type. But $L(7, 1) \not\simeq_S L(7, 2)$ [6, §31] and so $M/\{F_1\} \not\cong M/\{F_2\}$. So H cannot exist and f_1 and f_2 are not equivalent.

Counterexample 2. Let S^3, f_1, f_2 be as in Counterexample 1. Let $T = S^3/\{f_1^i(0, e^{i\pi(0)})\}$. Note that $f_1^i(0, e^{i\pi(0)}) = f_2^i(0, e^{i\pi(0)})$ for all i. Let $q \colon S^3 \to T$ be the quotient map. From the definition of T, f_1, and f_2, $k_1 = qf_1q^{-1}$ and $k_2 = qf_2q^{-1}$ are well-defined semifree period 7 homeomorphisms of the CW-complex T.

Let $*$ denote the unique fixed point of the actions k_1 and k_2. Clearly k_1 and k_2 revolve trivially at $*$. Write $Q = \prod_{i=1}^{\infty} D_i$ where each D_i is a two disk centered at the origin. Let $r_i\colon D_i^2 \to D_i^2$ be the standard rotation of D_i through $2\pi/7$ radians. Define $R\colon Q \to Q$ by $R(x_1, x_2, \ldots) = (r_1(x_1), r_2(x_2), \ldots)$ where $x_i \in D_i$ for all i. By a theorem of West [11], $T \times Q$ is a Q manifold. Define $K_i = k_i \times R\colon T \times Q \to T \times Q$, $i = 1, 2$. Each K_i is a semifree period 7 homeomorphism revolving trivially around the same unique fixed point. $J = qgq^{-1} \times R$ is clearly a homotopy between K_1 and K_2 where g is the homotopy from the previous example.

Now suppose K_1 was equivalent to K_2, that is there exists P such that $K_2 = PK_1P^{-1}$. Then P would induce a homeomorphism $\bar{P}$ such that the following diagram would commute.

$$\begin{tikzcd}
T \times Q \arrow[rr, "S_1"] \arrow[dr, "P"] \arrow[dd, "K_1"'] & & T \times Q/\{K_1\} \arrow[dr, "\bar{P}"] \arrow[dd, "\mathrm{id}" near start] & \\
& T \times Q \arrow[rr, "S_2" near start] \arrow[dd, "K_2" near start] & & T \times Q/\{K_2\} \arrow[dd, "\mathrm{id}"] \\
T \times Q \arrow[rr, "S_1" near end] \arrow[dr, "P"] & & T \times Q/\{K_1\} \arrow[dr, "\bar{P}"] & \\
& T \times Q \arrow[rr, "S_2"] & & T \times Q/\{K_2\}
\end{tikzcd}$$

S_1 and S_2 are the naturally induced quotient maps. This would imply that $T \times Q/\{K_1\} \cong T \times Q/\{K_2\}$. But it will now be shown that $(T \times Q/\{K_1\}) \times Q \cong L(7, 1) \times Q$ and $(T \times Q/\{K_2\}) \times Q \cong L(7, 2) \times Q$, and so P cannot exist.

Define

$$T_0 = T/\{k_1\} \times (0, 0, \ldots) \times Q,$$

and

$$T_n = \frac{T \times \prod_{i=1}^{n} D_i}{k_1 \times \prod_1^n r_i} \times (0_{n+1}, 0_{n+2}, \ldots) \times Q$$

for all $n \geqslant 1$.

Define $C_n\colon T_{n+1} \to T_n$ by

$$C_n = \mathrm{id}_T \times \prod_{i=1}^{n} \mathrm{id}_{D_i} \times C_{n+1} \times \mathrm{id}$$

where C_{n+1} is induced by the radial collapse of D_{n+1} to the origin. By the previously mentioned theorem of West, T_n is a Q manifold. Each C_n is a CE-mapping and hence, by Chapman's CE-mapping Theorem [3], [4] (see [8] for a short proof), each C_i is a near homeomorphism. So by Brown's Inverse Limit Theorem [2], proj $\lim\{T_i, C_i\} \cong T_0$. It is easy to put a metric on $T \times Q$ so that each C_i is within 2^{-i} of the identity map. Since each C_i is nonexpansive, the family $\{C_n \circ \cdots \circ C_{n+j}\} \cup_{j=0}^{\infty}$ is a uniformly equicontinuous family of maps. Certainly $\bigcup_1^\infty T_i \times Q$ is dense in $((T \times Q)/\{K_1\}) \times Q$. So by a lemma of R. Schori and D. Curtis [10], $((T \times Q)/\{K_1\}) \times Q \cong$

proj $\lim\{T_i, C_i\}$. So $((T \times Q)/\{K_1\}) \times Q \cong T_0$. But $T_0 = (T/\{k_1\}) \times (0, 0, \ldots) \times Q \cong L(7, 1) \times Q$. The proof that $((T \times Q)/\{K_2\}) \times Q \cong L(7, 2) \times Q$ is obviously identical. So K_1 and K_2 are not equivalent. $\square$

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