

A TWO-CARDINAL THEOREM AND A COMBINATORIAL THEOREM

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ABSTRACT. We prove a new two-cardinal theorem, e.g. $(\aleph_\omega, \aleph_0) \rightarrow (2^{\aleph_0}, \aleph_0)$.
 For this we prove a combinatorial theorem.

This is a sequel of Shelah [S1], and solves the main problem there. This problem also appears in Chang and Keisler [CK] and Friedman [Fr, Problem 30]. Our result is:

THEOREM 1. (A) *If for every $n < \omega$, the first order theory T has a model type $(\aleph_{\alpha+n}, \aleph_\alpha)$ then whenever $|T| \leq \mu < \lambda < \text{Ded}^* \mu$, T has a model type (λ, μ) .*

(B) *If $\aleph_{\alpha+\omega} < \text{Ded}^* \aleph_\alpha$ then $(\aleph_{\alpha+\omega}, \aleph_\alpha)$ is \aleph_α -compact and is complete.*

REMARK. $\text{Ded}^* \mu$ is the first cardinal χ such that no tree with $\leq \mu$ nodes has $\geq \chi$ branches of the same height. Note that $\text{Ded}^* \aleph_0 = (2^{\aleph_0})^+$, for every $\lambda \lambda^+ < \text{Ded}^* \lambda \leq (2^\lambda)^+$, and it is consistent with ZFC that $\text{Ded}^* \aleph_1 \leq 2^{\aleph_1}$.

This leads to many conjectures whose difficulty is not known to me; a sample is:

CONJECTURE 2. (A) $(\aleph_{\alpha+\omega+\omega}, \aleph_{\alpha+\omega}, \aleph_\alpha) \rightarrow (\lambda, \mu, \chi)$ whenever $\chi < \mu < \lambda < \text{Ded}^* \chi$.

(B) *If a countable theory T has a λ -like model, λ a limit cardinal, and $|T| \leq \mu < \lambda_1 < \text{Ded}^* \mu$, λ_1 a singular cardinal then T has a λ_1 -like model. If λ is M_ω -Mahlo weakly inaccessible cardinal, we can remove the singularity of λ_1 .*

(C) *If $\psi \in L_{\omega_1, \omega}$ has a model of cardinality \aleph_{ω_1} , then ψ has a model of cardinality 2^{\aleph_0} .*

NOTATION. Let I denote a well-ordered set. A (λ, n) -box B is $\prod_{l < n} I_l$ where I_l has order type λ ; λ, μ, χ denote infinite cardinals, elements of boxes will be denoted by η, τ, ν , and $\eta = \langle \eta(0), \dots, \eta(n-1) \rangle$. For a (λ, n) -box B , and $\eta_l \in B$ ($l < n$) we say $\langle \eta_0, \dots, \eta_{n-1} \rangle$ is proper for B if $k \neq l < n \Rightarrow \eta_k(l) < \eta_l(l)$.

Let $\lambda^{+0} = \lambda$, $\lambda^{+(k+1)} = (\lambda^{+k})^+ =$ the successor of λ^{+k} .

A B -indexed set is $\{a_\eta : \eta \in B\}$ such that $\eta \neq \tau \rightarrow a_\eta \neq a_\tau$. Under those conditions $\langle a_{\eta_0}, \dots \rangle$ is proper, iff $\langle \eta_0, \dots \rangle$ is proper. A (λ, n) -indexed set is a B -indexed set for some (λ, n) -box B .

LEMMA 3. *Suppose that $f_\alpha : (\lambda^+)^2 \rightarrow \lambda$ for each $\alpha < \lambda$. Then there exist*

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$s, t < \lambda^+$ such that for each $\alpha < \lambda$ there exist a, b, c, d for which $s < a < b < \lambda^+, t < c < d < \lambda^+$ and $f_\alpha(s, t) = f_\alpha(a, d) = f_\alpha(b, c)$.

PROOF. For every $u < \lambda^+$ let $t_u < \lambda^+$ be such that whenever $\alpha < \lambda$ and $t \geq t_u$, then

$$|\{v < \lambda^+ : f_\alpha(u, t) = f_\alpha(u, v)\}| = \lambda^+.$$

Let $X_u = \{(\alpha, \beta) : f_\alpha(u, t) = \beta \text{ for some } t \geq t_u\}$. Now let $s < \lambda^+$ be such that whenever $u \geq s$ and $(\alpha, \beta) \in X_u$, then

$$|\{v < \lambda^+ : (\alpha, \beta) \in X_v\}| = \lambda^+.$$

Let $t = t_s$. It is clear that this s and t work.

LEMMA 4. Let $f: A^n \rightarrow \lambda$ where A is any $(\lambda^+, n + 1)$ -indexed set and $k < n$. Then there is a (λ, n) -indexed set $A^* \subset A$ such that:

(*) For any proper sequence $\langle a_0, \dots, a_{n-1} \rangle$ from A^* there is a proper sequence $\langle b_0, \dots, b_n \rangle$ from A such that

$$f(a_0, \dots, a_{n-1}) = f(b_0, \dots, b_{n-1}) = f(b_0, \dots, b_{k-1}, b_n, b_{k+1}, \dots, b_{n-1}).$$

PROOF. Let A be a B -indexed set where $B = \prod_{l < n+1} I_l$ and $A = \{a_\eta : \eta \in B\}$. For notational simplicity let $k = n - 1$ and each $I_l = \lambda^+$. Now we define $\langle s_\alpha : \alpha < \lambda \rangle$ and $\langle t_\alpha : \alpha < \lambda \rangle$ by induction on α such that:

- (i) $s_\alpha, t_\alpha < \lambda^+$.
- (ii) $\langle s_\alpha : \alpha < \lambda \rangle$ and $\langle t_\alpha : \alpha < \lambda \rangle$ are increasing.
- (iii) Whenever $\eta_0, \dots, \eta_{n-2} \in B$ and $\tau \in \lambda^{n-1}$ are such that for each $i, l < n - 1$ there is $\beta < \alpha$ such that $\eta_i(l) < \lambda, \eta_i(n - 1) = s_\beta$ and $\eta_i(n) = t_\beta$, then there are a, b, c, d such that $s_\alpha < a < b < \lambda^+, t_\alpha < c < d < \lambda^+$ and

$$\begin{aligned} f(a_{\eta_0}, \dots, a_{\eta_{n-2}}, a_{\tau \wedge \langle s_\alpha, t_\alpha \rangle}) &= f(a_{\eta_0}, \dots, a_{\eta_{n-2}}, a_{\tau \wedge \langle a, d \rangle}) \\ &= f(a_{\eta_0}, \dots, a_{\eta_{n-2}}, a_{\tau \wedge \langle b, c \rangle}). \end{aligned}$$

Suppose we have defined s_β and t_β for $\beta < \alpha$. For each $\eta_0, \dots, \eta_{n-2}, \tau$ satisfying the conditions of (iii), there is a function $g: (\lambda^+)^2 \rightarrow \lambda$ defined by

$$g(x, y) = f(a_{\eta_0}, \dots, a_{\eta_{n-2}}, a_{\tau \wedge \langle x, y \rangle}).$$

There are $\leq \lambda$ such functions g . So we can apply Lemma 3 to get s_α, t_α such that $s_\alpha > t_\beta$ and $t_\alpha > t_\beta$ for each $\beta < \alpha$, and for each such g there are a, b, c, d such that $s_\alpha < a < b < \lambda^+, t_\alpha < c < d < \lambda^+$ and $g(s_\alpha, t_\alpha) = g(a, d) = g(b, c)$.

Now we define the (λ, n) -indexed set A^* . For each $\tau \in \lambda^{n-1}$ let $b_{\tau \wedge \langle \alpha \rangle} = a_{\tau \wedge \langle s_\alpha, t_\alpha \rangle}$. Then let $A^* = \{b_\eta : \eta \in \lambda^n\}$. Now it is easy to check that (*) holds.

THEOREM 5. Let $f_l: (\lambda^{+n})^l \rightarrow \lambda$ whenever $0 < l \leq n$, and let $h: (n + 1) \rightarrow n$ be such that $h(l) < l$ whenever $0 < l \leq n$. Then there are distinct $a_0, \dots, a_n < \lambda^{+n}$ such that

$$f_l(a_0, \dots, a_{l-1}) = f_l(a_0, \dots, a_{h(l)-1}, a_l, a_{h(l)+1}, \dots, a_{l-1})$$

whenever $0 < l \leq n$.

PROOF. We let λ^{+n} be $(\lambda^{+n}, n + 1)$ -indexed. Now we prove the theorem by induction on n . For $n = 0$ there is nothing to prove. For $n + 1$ we use Lemma 4, and the induction hypothesis on A^* .

PROOF OF THEOREM 1. Clear from [S1, §3] and Lemma 4.

REMARKS. (1) The following theorem is clear.

THEOREM . *If every finite subset of T has, for each n , a model of type $(\lambda_m, \dots, \lambda_0)$ where $\lambda_0^{+n} < \lambda_1, [(\lambda_{l+1})^{(\lambda_l^{+n})}]^{+n} < \lambda_{l+2}$ ($l < n - 1$) and $|T| \leq \mu_0 \leq \mu_1 \leq \dots \leq \mu_m < \text{Ded}^* \mu_0$ then T has a model of type (μ_m, \dots, μ_0) . (The parallel theorem in [S1] was noted by Papageorgiou.)*

(2) We can prove the main theorem of [S1] in a way similar to the proof here.

(3) In the notation of Shelah [S2, §3], we have proved in Lemma 4 that for $m = 2^n, r < \omega, \lambda^{+m} \xrightarrow{w} (n)_\lambda^r$. This answers positively question 3 from [S2]. But it is still unknown whether we have the best results.

(4) Halperin and Levi [H Le] used an indiscernibility similar to the one used in [S1], and Halperin and Lauchli proved the necessary combinatorial theorem. We have not succeeded in generalizing their proof. However, we can use our method to prove a weaker variant of their theorem, which is sufficient to prove that if $T \subset T_1, |T_1| = \aleph_0, T$ is complete and there are $> \aleph_0$ complete $L(T)$ -types consistent with $T, \lambda > \aleph_0$ then T has $\geq \min\{2^\lambda, 2^{2^{\aleph_0}}\}$ nonisomorphic $L(T)$ -reducts of models of T_1 . (This will appear in [S3].)

(5) To see the connection note that by [S3] it follows by Theorem 5 that

THEOREM . *Let $f_j: (\lambda^{+n})^l \rightarrow \lambda$ whenever $0 < l \leq n, n = 2^m - 1$. Then there are distinct $a_\eta < \lambda^{+n}$ for $\eta \in 2^m$ (i.e. η is a sequence of ones and zeros of length m) such that: if $k < m, 0 < l \leq n, \tau_1, \dots, \tau_l$ are distinct members of 2^k , and for $1 \leq i \leq l, 0 \leq j \leq 1, \eta_i^j \in 2^m$ and τ_i is an initial segment of η_i^j then $f_l(\eta_1^0, \dots, \eta_l^0) = f_l(\eta_1^1, \dots, \eta_l^1)$.*

(6) In Lemma 4 and Theorem 5 we can consider μ such functions, provided that $\mu \leq x, \lambda^\mu = \lambda$, resp.

(7) We can use only a (B, n) -indexed set for a fixed n in (4), but then in Remark (3) m will become bigger.

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