

ON THE SCHNIRELMANN DENSITY OF THE k -FREE INTEGERS

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ABSTRACT. Let $Q_k(n)$ be the number of k -free integers $\leq n$ and $d(Q_k)$ the Schnirelmann density of the k -free integers. If $k \geq 5$, it is shown that $Q_k(n)/n = d(Q_k)$ for some n satisfying $6^k/2 \leq n < 6^k$ and certain other properties, and that

$$d(Q_k) \geq 1 - 2^{-k} - 3^{-k} - 5^{-k} + (3^{-k} + 2 \cdot 5^{-k})(6^k - 3^k + 1)^{-1}.$$

$d(Q_k)$ and the n for which $Q_k(n)/n = d(Q_k)$ are found for $7 \leq k \leq 12$.

1. Introduction. We denote the set of positive k -free integers by Q_k and the number of integers $\leq x$ and Q_k by $Q_k(x)$. The Schnirelmann density of Q_k is

$$d(Q_k) = \inf_{n \geq 1} Q_k(n)/n.$$

Here, and throughout this note, n denotes a positive integer.

K. Rogers [3] proved that

$$(1) \quad d(Q_2) = \frac{53}{88} \quad \text{and} \quad \frac{Q_2(n)}{n} = d(Q_2) \quad \text{iff } n = 176,$$

and R. L. Duncan [1] that

$$(2) \quad d(Q_k) > 1 - \sum_{\text{prime } p > 0} \frac{1}{p^k}.$$

More recently, R. C. Orr [2] proved that

$$(3) \quad d(Q_3) = \frac{157}{189} \quad \text{and} \quad \frac{Q_3(n)}{n} = d(Q_3) \quad \text{iff } n = 378,$$

$$(4) \quad d(Q_4) = \frac{145}{157} \quad \text{and} \quad \frac{Q_4(n)}{n} = d(Q_4) \quad \text{iff } n = 2512,$$

$$(5) \quad d(Q_5) = \frac{3055}{3168} \quad \text{and} \quad \frac{Q_5(n)}{n} = d(Q_5) \quad \text{iff } n = 3168 \text{ or } 6336,$$

$$(6) \quad d(Q_6) = \frac{6165}{6272} \quad \text{and} \quad \frac{Q_6(n)}{n} = d(Q_6) \quad \text{iff } n = 31360,$$

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and

$$(7) \quad \begin{aligned} & \text{if } k \geq 5, Q_k(n)/n = d(Q_k) \text{ for some } n \text{ satisfying} \\ & 5^k \leq n < 6^k, \text{ but for no } n < 5^k \text{ or } \geq 6^k. \end{aligned}$$

In this note we use Orr's and Rogers's results to improve Duncan's inequality to

$$(8) \quad d(Q_k) > 1 - 2^{-k} - 3^{-k} - 5^{-k},$$

and to show that

$$(9) \quad \begin{aligned} & \text{if } k \geq 5, Q_k(n)/n = d(Q_k) \text{ for some } n \text{ satisfying} \\ & 6^k/2 \leq n < 6^k. \end{aligned}$$

We next use (9) to prove

THEOREM 1. *If $k \geq 5$ then $Q_k(n)/n = d(Q_k)$ for some n which is such that $6^k/2 \leq n < 6^k$ and either (i) n is a multiple of 3^k or 5^k , or (ii) n is a multiple of 2^k and there is a multiple of 3^k or 5^k between $n - 2^k$ and n .*

We then use this theorem to obtain, for $k \geq 5$, the refinement

$$(10) \quad d(Q_k) \geq 1 - 2^{-k} - 3^{-k} - 5^{-k} + (3^{-k} + 2 \cdot 5^{-k})(6^k - 3^k + 1)^{-1}$$

of the inequality (8), and also to find, for $7 \leq k \leq 12$, $d(Q_k)$ and the n for which $Q_k(n)/n = d(Q_k)$.

2. Proof of (8). For $k \geq 5$, we use Orr's result (7). For any $n < 6^k$,

$$Q_k(n) = n - \left[\frac{n}{2^k} \right] - \left[\frac{n}{3^k} \right] - \left[\frac{n}{5^k} \right] > n - \frac{n}{2^k} - \frac{n}{3^k} - \frac{n}{5^k},$$

since no $n < 6^k$ is divisible by more than one of 2^k , 3^k and 5^k . Thus, using (7), we have (8) for $k \geq 5$.

To complete the proof we have merely to check (8) for $k \leq 4$, using Rogers's and Orr's results (1), (3) and (4).

3. Proof of (9). By Orr's result (7), if $k \geq 5$, $d(Q_k) = Q_k(n)/n$ for some $n < 6^k$. If this $n \geq 6^k/2$ then (9) is proved. If not, let $m = [(6^k - 1)/n]$. Then $m > 1$ and

$$\begin{aligned} Q_k(mn) &= mn - [mn/2^k] - [mn/3^k] - [mn/5^k] \\ &\leq mn - m[n/2^k] - m[n/3^k] - m[n/5^k] = mQ_k(n), \end{aligned}$$

and so $Q_k(mn)/mn \leq Q_k(n)/n$. Thus (9) is proved, since $mn \geq 6^k/2$, clearly.

REMARK. A similar proof shows that, if k , n and m are as above, then $Q_k(rn)/rn = d(Q_k)$ for $1 \leq r \leq m$. It is easy to see that

$$mn > m(6^k - 1)/(m + 1).$$

Thus, if $Q_k(n)/n = d(Q_k)$ for some $n < 6^k/2$, then $Q_k(n)/n = d(Q_k)$ for some $n \geq (2/3)6^k$.

We note that, for any $k \geq 5$, if the n , for which $6^k/2 \leq n < 6^k$ and $Q_k(n)/n = d(Q_k)$, are known, then it is possible to find the n' for which $5^k \leq n' < 6^k/2$ and $Q_k(n')/n' = d(Q_k)$, since, for some such n, n' must be a multiple of $n/(n, Q_k(n))$ and n of n' , and since $Q_k(n)/n = Q_k(n')/n'$ implies that $[n/a^k] = (n/n')[n'/a^k]$ for $a = 2, 3$ and 5 .

4. Proof of Theorem 1. Let $k \geq 5$, and n_0, n_1 be such that no m ($n_0 < m < n_1 < 6^k$) is a multiple of 2^k or 3^k or 5^k . Then it is easy to see that $Q_k(m)/m > Q_k(n_0)/n_0$. It is also easy to see that if n_0, n_1 are multiples of 2^k such that no m ($6^k/2 \leq n_0 < m < n_1 < 6^k$) is a multiple of 3^k or 5^k , then $Q_k(n_1)/n_1 > Q_k(n_0)/n_0$. Hence we have Theorem 1.

5. A refinement of (8). We use Theorem 1. If (i) is satisfied and $3^k | n$, then

$$\begin{aligned} Q_k(n)n^{-1} &= n^{-1}\{n - [n \cdot 2^{-k}] - [n \cdot 3^{-k}] - [n \cdot 5^{-k}]\} \\ &= 1 - 2^{-k} - 3^{-k} - 5^{-k} + (\alpha \cdot 2^{-k} + \beta \cdot 5^{-k})n^{-1}, \end{aligned}$$

where α, β are the remainders when n is divided by $2^k, 5^k$, respectively. Since $n < 6^k$ and no $n < 6^k$ is divisible by more than one of $2^k, 3^k$ and 5^k , it follows that

$$\begin{aligned} Q_k(n)n^{-1} &\geq 1 - 2^{-k} - 3^{-k} - 5^{-k} + n^{-1} \min\{2^{-k} + 2 \cdot 5^{-k}, 2 \cdot 2^{-k} + 5^{-k}\} \\ &\geq 1 - 2^{-k} - 3^{-k} - 5^{-k} + (2^{-k} + 2 \cdot 5^{-k})(6^k - 1)^{-1}. \end{aligned}$$

Similarly, if (i) is satisfied and $5^k | n$, then

$$Q_k(n)n^{-1} \geq 1 - 2^{-k} - 3^{-k} - 5^{-k} + (2^{-k} + 2 \cdot 3^{-k})(6^k - 1)^{-1}.$$

If (ii) is satisfied, then, similarly,

$$\begin{aligned} Q_k(n)n^{-1} &\geq 1 - 2^{-k} - 3^{-k} - 5^{-k} \\ &\quad + \min\{(3^{-k} + 2 \cdot 5^{-k})(6^k - 3^k + 1)^{-1}, \\ &\quad (2 \cdot 3^{-k} + 5^{-k})(6^k - 1)^{-1}\}, \end{aligned}$$

since $6^k - 3^k + 1$ is the largest $n < 6^k$ and $\equiv 1 \pmod{3^k}$.

The inequality (10) now follows since it can be shown that

$$(3^{-k} + 2 \cdot 5^{-k})(6^k - 3^k + 1)^{-1} \leq (2 \cdot 3^{-k} + 5^{-k})(6^k - 1)^{-1}$$

and

$$2 \cdot 3^{-k} + 5^{-k} \leq 2^{-k} + 2 \cdot 5^{-k} \leq 2^{-k} + 2 \cdot 3^{-k}.$$

We have proved the refinement (10) of (8) for $k \geq 5$. It is true for $k = 3$ also, but not for $k = 2$ or 4 . We thus have

THEOREM 2. *The inequality (10) holds for $k = 3$ and $k \geq 5$, but not for $k = 2$ or 4 .*

6. **Computation of $d(Q_k)$ for $k \geq 7$.** We can compute these, using Theorem 1; the number of computations of $Q_k(n)/n$ needed to compute $d(Q_k)$ is, approximately, $2^k + (6/5)^k$.

For $7 \leq k \leq 12$, the values of $d(Q_k)$ are as follows:

$$\begin{aligned} d(Q_7) &= \frac{234331}{236288}, & d(Q_8) &= \frac{1169758}{1174528}, \\ d(Q_9) &= \frac{7798488}{7814151}, & d(Q_{10}) &= \frac{48785015}{48833536}, \\ d(Q_{11}) &= \frac{292856489}{293001216}, & d(Q_{12}) &= \frac{1709225206}{1709645824}. \end{aligned}$$

For each of these k , there is only one n such that $Q_k(n)/n = d(Q_k)$, and this n is given as the denominator in the value of $d(Q_k)$.

The following table gives, for $2 \leq k \leq 12$, the values, correct to ten decimal places, of the Schnirelmann density $d(Q_k)$ and the asymptotic density $\delta(Q_k) = \lim_{n \rightarrow \infty} Q_k(n)/n = 1/\zeta(k)$ of Q_k .

k	$d(Q_k)$	$\delta(Q_k)$
2	.6022727273	.6079271019
3	.8306878307	.8319073726
4	.9235668790	.9239384029
5	.9643308081	.9643873404
6	.9829400510	.9829525923
7	.9917177343	.9917198558
8	.9959387941	.9959392011
9	.9979955596	.9979956327
10	.9990064000	.9990064131
11	.9995060532	.9995060555
12	.9997539736	.9997539740

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