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ON THE SCHNIRELMANN DENSITY OF THE k-FREE INTEGERS

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ABSTRACT. Let $Q_k(n)$ be the number of k-free integers $\leqslant n$ and $d(Q_k)$ the Schnirelmann density of the k-free integers. If $k \geqslant 5$, it is shown that $Q_k(n)/n = d(Q_k)$ for some n satisfying $6^k/2 \leqslant n < 6^k$ and certain other properties, and that

$$d(Q_k) \geqslant 1 - 2^{-k} - 3^{-k} - 5^{-k} + (3^{-k} + 2 \cdot 5^{-k})(6^k - 3^k + 1)^{-1}.$$

$d(Q_k)$ and the n for which $Q_k(n)/n = d(Q_k)$ are found for $7 \leqslant k \leqslant 12$.

1. **Introduction.** We denote the set of positive k-free integers by Q_k and the number of integers $\leqslant x$ and Q_k by $Q_k(x)$. The Schnirelmann density of Q_k is

$$d(Q_k) = \inf_{n \geqslant 1} Q_k(n)/n.$$

Here, and throughout this note, n denotes a positive integer.

K. Rogers [3] proved that

$$d(Q_2) = \frac{53}{88} \quad \textit{and} \quad \frac{Q_2(n)}{n} = d(Q_2) \quad \textit{iff } n = 176, \tag{1}$$

and R. L. Duncan [1] that

$$d(Q_k) > 1 - \sum_{\textit{prime } p > 0} \frac{1}{p^k}. \tag{2}$$

More recently, R. C. Orr [2] proved that

$$d(Q_3) = \frac{157}{189} \quad \textit{and} \quad \frac{Q_3(n)}{n} = d(Q_3) \quad \textit{iff } n = 378, \tag{3}$$

$$d(Q_4) = \frac{145}{157} \quad \textit{and} \quad \frac{Q_4(n)}{n} = d(Q_4) \quad \textit{iff } n = 2512, \tag{4}$$

$$d(Q_5) = \frac{3055}{3168} \quad \textit{and} \quad \frac{Q_5(n)}{n} = d(Q_5) \quad \textit{iff } n = 3168 \textit{ or } 6336, \tag{5}$$

$$d(Q_6) = \frac{6165}{6272} \quad \textit{and} \quad \frac{Q_6(n)}{n} = d(Q_6) \quad \textit{iff } n = 31360, \tag{6}$$

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and

$$\text{(7)} \qquad \begin{array}{l} \textit{if } k \geqslant 5,\ Q_k(n)/n = d(Q_k) \textit{ for some } n \textit{ satisfying} \\ 5^k \leqslant n < 6^k, \ \textit{but for no } n < 5^k \textit{ or } \geqslant 6^k. \end{array}$$

In this note we use Orr's and Rogers's results to improve Duncan's inequality to

$$\text{(8)} \qquad d(Q_k) > 1 - 2^{-k} - 3^{-k} - 5^{-k},$$

and to show that

$$\text{(9)} \qquad \begin{array}{l} \textit{if } k \geqslant 5,\ Q_k(n)/n = d(Q_k) \textit{ for some } n \textit{ satisfying} \\ 6^k/2 \leqslant n < 6^k. \end{array}$$

We next use (9) to prove

THEOREM 1. *If $k \geqslant 5$ then $Q_k(n)/n = d(Q_k)$ for some n which is such that $6^k/2 \leqslant n < 6^k$ and either* (i) *n is a multiple of 3^k or 5^k, or* (ii) *n is a multiple of 2^k and there is a multiple of 3^k or 5^k between $n - 2^k$ and n.*

We then use this theorem to obtain, for $k \geqslant 5$, the refinement

$$\text{(10)} \quad d(Q_k) \geqslant 1 - 2^{-k} - 3^{-k} - 5^{-k} + (3^{-k} + 2 \cdot 5^{-k})(6^k - 3^k + 1)^{-1}$$

of the inequality (8), and also to find, for $7 \leqslant k \leqslant 12$, $d(Q_k)$ and the n for which $Q_k(n)/n = d(Q_k)$.

2. **Proof of (8).** For $k \geqslant 5$, we use Orr's result (7). For any $n < 6^k$,

$$Q_k(n) = n - \left[\frac{n}{2^k}\right] - \left[\frac{n}{3^k}\right] - \left[\frac{n}{5^k}\right] > n - \frac{n}{2^k} - \frac{n}{3^k} - \frac{n}{5^k},$$

since no $n < 6^k$ is divisible by more than one of 2^k, 3^k and 5^k. Thus, using (7), we have (8) for $k \geqslant 5$.

To complete the proof we have merely to check (8) for $k \leqslant 4$, using Rogers's and Orr's results (1), (3) and (4).

3. **Proof of (9).** By Orr's result (7), if $k \geqslant 5$, $d(Q_k) = Q_k(n)/n$ for some $n < 6^k$. If this $n \geqslant 6^k/2$ then (9) is proved. If not, let $m = [(6^k - 1)/n]$. Then $m > 1$ and

$$\begin{aligned} Q_k(mn) &= mn - [mn/2^k] - [mn/3^k] - [mn/5^k] \\ &\leqslant mn - m[n/2^k] - m[n/3^k] - m[n/5^k] = mQ_k(n), \end{aligned}$$

and so $Q_k(mn)/mn \leqslant Q_k(n)/n$. Thus (9) is proved, since $mn \geqslant 6^k/2$, clearly.

REMARK. A similar proof shows that, if k, n and m are as above, then $Q_k(rn)/rn = d(Q_k)$ for $1 \leqslant r \leqslant m$. It is easy to see that

$$mn > m(6^k - 1)/(m + 1).$$

Thus, if $Q_k(n)/n = d(Q_k)$ for some $n < 6^k/2$, then $Q_k(n)/n = d(Q_k)$ for some $n \geqslant (2/3)6^k$.

We note that, for any $k \geqslant 5$, if the n, for which $6^k/2 \leqslant n < 6^k$ and $Q_k(n)/n = d(Q_k)$, are known, then it is possible to find the n' for which $5^k \leqslant n' < 6^k/2$ and $Q_k(n')/n' = d(Q_k)$, since, for some such n, n' must be a multiple of $n/(n, Q_k(n))$ and n of n', and since $Q_k(n)/n = Q_k(n')/n'$ implies that $[n/a^k] = (n/n')[n'/a^k]$ for $a = 2$, 3 and 5.

4. **Proof of Theorem 1.** Let $k \geqslant 5$, and n_0, n_1 be such that no m ($n_0 < m < n_1 < 6^k$) is a multiple of 2^k or 3^k or 5^k. Then it is easy to see that $Q_k(m)/m > Q_k(n_0)/n_0$. It is also easy to see that if n_0, n_1 are multiples of 2^k such that no m ($6^k/2 \leqslant n_0 < m < n_1 < 6^k$) is a multiple of 3^k or 5^k, then $Q_k(n_1)/n_1 > Q_k(n_0)/n_0$. Hence we have Theorem 1.

5. **A refinement of (8).** We use Theorem 1. If (i) is satisfied and $3^k|n$, then

$$\begin{aligned} Q_k(n)n^{-1} &= n^{-1}\{n - [n \cdot 2^{-k}] - [n \cdot 3^{-k}] - [n \cdot 5^{-k}]\} \\ &= 1 - 2^{-k} - 3^{-k} - 5^{-k} + (\alpha \cdot 2^{-k} + \beta \cdot 5^{-k})n^{-1}, \end{aligned}$$

where α, β are the remainders when n is divided by 2^k, 5^k, respectively. Since $n < 6^k$ and no $n < 6^k$ is divisible by more than one of 2^k, 3^k and 5^k, it follows that

$$\begin{aligned} Q_k(n)n^{-1} &\geqslant 1 - 2^{-k} - 3^{-k} - 5^{-k} + n^{-1}\min\{2^{-k} + 2 \cdot 5^{-k}, 2 \cdot 2^{-k} + 5^{-k}\} \\ &\geqslant 1 - 2^{-k} - 3^{-k} - 5^{-k} + (2^{-k} + 2 \cdot 5^{-k})(6^k - 1)^{-1}. \end{aligned}$$

Similarly, if (i) is satisfied and $5^k|n$, then

$$Q_k(n)n^{-1} \geqslant 1 - 2^{-k} - 3^{-k} - 5^{-k} + (2^{-k} + 2 \cdot 3^{-k})(6^k - 1)^{-1}.$$

If (ii) is satisfied, then, similarly,

$$\begin{aligned} Q_k(n)n^{-1} \geqslant\ & 1 - 2^{-k} - 3^{-k} - 5^{-k} \\ & + \min\{(3^{-k} + 2 \cdot 5^{-k})(6^k - 3^k + 1)^{-1}, \\ & \qquad (2 \cdot 3^{-k} + 5^{-k})(6^k - 1)^{-1}\}, \end{aligned}$$

since $6^k - 3^k + 1$ is the largest $n < 6^k$ and $\equiv 1 \pmod{3^k}$.

The inequality (10) now follows since it can be shown that

$$(3^{-k} + 2 \cdot 5^{-k})(6^k - 3^k + 1)^{-1} \leqslant (2 \cdot 3^{-k} + 5^{-k})(6^k - 1)^{-1}$$

and

$$2 \cdot 3^{-k} + 5^{-k} \leqslant 2^{-k} + 2 \cdot 5^{-k} \leqslant 2^{-k} + 2 \cdot 3^{-k}.$$

We have proved the refinement (10) of (8) for $k \geqslant 5$. It is true for $k = 3$ also, but not for $k = 2$ or 4. We thus have

THEOREM 2. *The inequality* (10) *holds for* $k = 3$ *and* $k \geqslant 5$, *but not for* $k = 2$ *or* 4.

6. **Computation of $d(Q_k)$ for $k \geqslant 7$.** We can compute these, using Theorem 1; the number of computations of $Q_k(n)/n$ needed to compute $d(Q_k)$ is, approximately, $2^k + (6/5)^k$.

For $7 \leqslant k \leqslant 12$, the values of $d(Q_k)$ are as follows:

$$d(Q_7) = \frac{234331}{236288}, \qquad d(Q_8) = \frac{1169758}{1174528},$$

$$d(Q_9) = \frac{7798488}{7814151}, \qquad d(Q_{10}) = \frac{48785015}{48833536},$$

$$d(Q_{11}) = \frac{292856489}{293001216}, \qquad d(Q_{12}) = \frac{1709225206}{1709645824}.$$

For each of these k, there is only one n such that $Q_k(n)/n = d(Q_k)$, and this n is given as the denominator in the value of $d(Q_k)$.

The following table gives, for $2 \leqslant k \leqslant 12$, the values, correct to ten decimal places, of the Schnirelmann density $d(Q_k)$ and the asymptotic density $\delta(Q_k) = \lim_{n\to\infty} Q_k(n)/n = 1/\zeta(k)$ of Q_k.

k	$d(Q_k)$	$\delta(Q_k)$
2	.6022727273	.6079271019
3	.8306878307	.8319073726
4	.9235668790	.9239384029
5	.9643308081	.9643873404
6	.9829400510	.9829525923
7	.9917177343	.9917198558
8	.9959387941	.9959392011
9	.9979955596	.9979956327
10	.9990064000	.9990064131
11	.9995060532	.9995060555
12	.9997539736	.9997539740

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