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WEAKLY STARLIKE MEROMORPHIC UNIVALENT FUNCTIONS. II

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ABSTRACT. A function $f(z)$ is said to be meromorphic weakly starlike if it has the form $f(z) = -\rho z F(Z)/(z-\rho)(1-\rho z)$ for $0 < \rho < 1$ where $F(z)$ is a member of Σ^*, the class of meromorphic, normalized starlike univalent functions. The coefficients of the power series expansion in $|z| < \rho$ of a meromorphic weakly starlike function are studied. The integral means of such functions are also discussed.

1. **Introduction**. Let $f(z)$ be meromorphic in the unit disk $\Delta = \{z \mid |z| < 1\}$ with a simple pole at $z = \rho$, $0 < \rho < 1$, and otherwise regular in Δ. $f(z)$ is said to be a member of $\Lambda(\rho)$ if and only if $f(0) = 1$ and there is a number ρ_1, $\rho < \rho_1 < 1$, such that with $z = re^{i\theta}$

$$\operatorname{Re}\left[zf'(z)/f(z)\right] < 0 \tag{1.1}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] d\theta = -1 \tag{1.2}$$

for $\rho_1 < r < 1$. Also let $\Lambda^*(\rho)$, $0 < \rho < 1$, be the class of functions $f(z)$ which have the representation

$$f(z) = -\rho z g(z)/\,(z-\rho)(1-\rho z) \tag{1.3}$$

where $g(z) = z^{-1} + a_0 + a_1 z + \ldots$ is a member of Σ^*, the class of meromorphic, normalized, starlike univalent functions [1].

The classes $\Lambda(\rho)$ and $\Lambda^*(\rho)$ have recently been studied by Libera and the author [8]. Functions in $\Lambda^*(\rho)$ are called weakly starlike meromorphic univalent functions since they are reciprocals of weakly starlike regular univalent functions introduced by Hummel [5], [6]. It was pointed out in [8] that functions in $\Lambda^*(\rho)$ are univalent and for each ρ, $\Lambda(\rho)$ is a subset of $\Lambda^*(\rho)$. Furthermore if $\rho > 1/2$, $\Lambda(\rho)$ is a proper subset of $\Lambda^*(\rho)$, but if

$$0 < \rho < (3 - 2\sqrt{2}\,)^{1/2} \approx \cdot 4,$$

then $\Lambda(\rho) = \Lambda^*(\rho)$.

If $f(z)$ is in $\Lambda^*(\rho)$ then it has a Taylor series expansion in $|z| < \rho$ of the

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form $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$. §2 of this paper concerns itself with the coefficients a_n of this expansion. In particular, if $f(z)$ is real for real values of z, we obtain precise upper and lower bounds on a_n for all n. §3 deals with integral means of a function in $\Lambda^*(\rho)$.

2. **Coefficient bounds.** Goodman [3] studied the class TM of functions $f(z)$ meromorphic in Δ with Taylor series in a neighborhood of the origin of the form $f(z) = z + \sum_{n=1}^{\infty} b_n z^n$ and which except at the poles satisfy the condition $\operatorname{Im} f(z) \operatorname{Im} z \geqslant 0$. Such functions are said to be meromorphic and typically real. Letting $TM(\rho)$ denote the subclass of TM of functions for which the poles ρ_j satisfy $|\rho_j| \geqslant \rho, j = 1, 2, 3, \ldots,$ Goodman proved [3] that for all n

$$|b_n| \leqslant (1 - \rho^{2n})/\rho^{n-1}(1 - \rho^2). \tag{2.1}$$

If we assume that $f(z)$ in $\Lambda^*(\rho)$ is real for real values of z and has the Taylor series expansion $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ for $|z| < \rho$, then since $f(z)$ is univalent it follows that the function $(f(z) - 1)/a_1$ is a member of $TM(\rho)$. We thus obtain from (2.1) the inequality,

$$|a_n| \leqslant \frac{1 - \rho^{2n}}{\rho^{n-1}(1 - \rho^2)} |a_1| \tag{2.2}$$

for all n. In [8], it was proven that $|a_1| \leqslant (1 + \rho)^2/\rho$. Combining this with (2.2) we obtain

$$|a_n| \leqslant \left(\frac{1 + \rho}{1 - \rho}\right)\left(\frac{1 - \rho^{2n}}{\rho^n}\right) \tag{2.3}$$

for all n. The interesting thing here however, is that under the assumption that $f(z)$ is real for real values of z, the coefficients a_n are necessarily positive and we can obtain precise lower bounds.

THEOREM 1. *If $f(z)$ is in $\Lambda^*(\rho)$ and real for real values of z and if $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ for $|z| < \rho$, then for all n*

$$\left(\frac{1 - \rho}{1 + \rho}\right)\left(\frac{1 - \rho^{2n}}{\rho^n}\right) \leqslant a_n \leqslant \left(\frac{1 + \rho}{1 - \rho}\right)\left(\frac{1 - \rho^{2n}}{\rho^n}\right). \tag{2.4}$$

The inequalities are sharp for each n. Equality is attained on the right side of (2.4) *for each n by the function $f(z) = -\rho(1 + z)^2/(z - \rho)(1 - \rho z)$ and on the left side of* (2.4) *for each n by the function $f(z) = -\rho(1 - z)^2/(z - \rho)(1 - \rho z)$.*

REMARK. It was proven in [8] that the function

$$f(z) = -\rho(1 - z)^2/(z - \rho)(1 - \rho z)$$

is actually in $\Lambda(\rho)$ for all ρ whereas the function

$$f(z) = -\rho(1 + z)^2/(z - \rho)(1 - \rho z)$$

is in $\Lambda(\rho)$ for $\rho \leqslant 1/2$ but is in $\Lambda^*(\rho) \setminus \Lambda(\rho)$ for $\rho > 1/2$. Thus the left-hand side of (2.4) is sharp in $\Lambda(\rho)$ for all ρ and the right side is sharp in $\Lambda(\rho)$ at least for $\rho < 1/2$.

PROOF OF THEOREM 1. It follows from (1.3) that the funtion

$$P(z) = zf'(z)/f(z) + \rho/(z-\rho) - \rho z/(1-\rho z). \tag{2.5}$$

has negative real part in Δ. Let

$$\frac{zf'(z)}{f(z)} = \sum_{n=1}^{\infty} b_n z^n \tag{2.6}$$

for $|z| < \rho$. Then from (2.5) we obtain

$$P(z) = -1 + \sum_{n=1}^{\infty}\left(b_n - \rho^n - \frac{1}{\rho^n}\right).$$

Since $P(z)$ has negative real part in Δ it follows that for each n

$$|b_n - \rho^n - 1/\rho^n| \leqslant 2. \tag{2.7}$$

Since by assumption $f(z)$ is real for real values of z, the coefficients b_n are real and we have from (2.7)

$$(1-\rho^n)^2/\rho^n \leqslant b_n \leqslant (1+\rho^n)^2/\rho^n \tag{2.8}$$

for all n. From (2.6) we have that

$$zf'(z) = \left(\sum_{n=1}^{\infty} b_n z^n\right) f(z) \tag{2.9}$$

for $|z| < \rho$. Comparing the coefficients of both sides of (2.9) we obtain

$$na_n = b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 \tag{2.10}$$

for all n. Thus in particular $a_1 = b_1 \geqslant (1-\rho)^2/\rho$ which is the left side of (2.4) for $n = 1$. Next suppose that $a_k \geqslant ((1-\rho)/(1+\rho))((1-\rho)^{2k}/\rho^k)$ for $k = 1, 2, \ldots, (n-1)$. Then from (2.4) we obtain

$$\begin{aligned}
na_n &\geqslant \frac{(1-\rho^n)^2}{\rho^n} + \sum_{k=1}^{n-1} \frac{(1-\rho)}{(1+\rho)} \frac{(1-\rho^{2k})(1-\rho^{n-k})^2}{\rho^n} \\
&= \frac{(1-\rho)}{\rho^n(1+\rho)}\left[(1-\rho^2)\left(\sum_{k=0}^{n-1}\rho^k\right)^2 + \sum_{k=1}^{n-1}(1-\rho^{2k})(1-\rho^{n-k})^2\right] \\
&= \frac{(1-\rho)}{\rho^n(1+\rho)}\Bigg[(1-\rho^2)\big(1 + 2\rho + 3\rho^2 + \cdots + n\rho^{n-1} + (n-1)\rho^n \\
&\qquad + (n-2)\rho^{n+1} + \cdots \rho^{2n-2}\big) + (n-1)(1-\rho^{2n}) \\
&\qquad - 2\sum_{k=1}^{n-1}\rho^{n-k} + 2\sum_{k=1}^{n-1}\rho^{n+k}\Bigg] \\
&= n((1-\rho)/(1+\rho))((1-\rho^{2n})/\rho^n).
\end{aligned}$$

Thus the left side of (2.4) follows for all n. Due to the remarks preceding Theorem 1, the right side of (2.4) now follows. In conclusion, we remark that the right side of (2.4) cannot be obtained for all n by making use of (2.10) and the right side of (2.8).

COROLLARY 1. *Let $f(z)$ be in $\Lambda^*(\rho)$. If $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ for $|z| < \rho$ and if a_n is real for $1 \leqslant n \leqslant N$, then*

$$a_n \geqslant ((1-\rho)/(1+\rho))((1-\rho^{2n})/\rho^n)$$

for $1 \leqslant n \leqslant N$. The inequalities are sharp.

PROOF. If $zf'(z)/f(z) = P(z) = \sum_{n=1}^{\infty} b_n z^n$ for $|z| < \rho$, then the assumptions of the corollary imply that b_n is real for $1 \leqslant n \leqslant N$. Thus (2.8) holds for $1 \leqslant n \leqslant N$. With this observation the proof proceeds as in Theorem 1.

COROLLARY 2. *Let $f(z)$ be in $\Lambda^*(\rho)$. If $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ for $|z| < \rho$ and if a_n is real for $1 \leqslant n \leqslant N$ then*

$$\operatorname{Re} a_{N+1} \geqslant ((1-\rho)/(1+\rho))((1-\rho^{2N+1})/\rho^{N+1}).$$

PROOF. As in the proof of Corollary 1 the coefficients b_n are real for $1 \leqslant n \leqslant N$. Thus taking real parts of both sides of (2.10) when $n = N + 1$, we can follow the argument in the proof of Theorem 1 to obtain the desired inequality.

In [3] Goodman conjectured that if $f(z)$ is meromorphic and univalent in Δ with a simple pole at $z = p$, $0 < \rho < 1$, and if $f(z) = z + \sum_{n=2}^{\infty} b_n z_n$ for $|z| < \rho$ then $|b_n| \leqslant (1-\rho^{2n})/\rho^{n-1}(1-\rho^2)$. Subsequently Jenkins [7] proved that if the Bieberbach conjecture is valid up to index N, then Goodman's conjecture is also valid up to index N. Using the fact that if $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ is a member of $\Lambda^*(\rho)$, $(f(z) - 1)/a_1$ is univalent in Δ with a simple pole at $z = \rho$ and using the fact that at $|a_1| \leqslant (1+\rho^2)/\rho$ [8], it follows that

$$|a_n| \leqslant ((1+\rho)/(1-\rho))((1-\rho^{2n})/\rho^n), \qquad n \leqslant 6. \tag{2.11}$$

With the assumption that $f(z)$ in $\Lambda^*(\rho)$ is real on the real axis we have obtained positive lower bounds on a_n for all n. It is natural then to ask whether there is in general a positive lower bound on $|a_n|$ for all $f(z)$ in $\Lambda^*(\rho)$.

3. **Integral means.** In this section we prove that the function

$$F(z) = -\rho(1+z)^2/(z-\rho)(1-\rho z)$$

has maximal L^p means in the class $\Lambda^*(\rho)$.

THEOREM 5. *Let $f(z)$ be in $\Lambda^*(\rho)$ and $F(z) = -\rho(1+z)^2/(z-\rho)(1-\rho z)$ then for any positive real number p and $r \neq \rho$,*

$$\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \leqslant \int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta. \tag{3.1}$$

PROOF. Since $f(z)$ is a member of $\Lambda^*(\rho)$ we can write

$$f(z) = -\rho z G(z)/(z-\rho)(1-\rho z) \tag{3.2}$$

where $G(z)$ is in Σ^*. Since $G(z)$ is in Σ^* there exists a nondecreasing function $m(t)$ on $[-\pi, \pi]$ with $\int_{-\pi}^{\pi} dm(t) = 1$ such that

$$G(z) = \frac{1}{z} \exp\left(2\int_{-\pi}^{\pi} \log(1 - e^{-it}z)\, dm(t)\right). \tag{3.3}$$

Thus from (3.2) and (3.3) we have

$$f(z) = \frac{-\rho}{(z-\rho)(1-\rho z)} \exp\left(2\int_{-\pi}^{\pi} \log(1 - e^{-it}z)\, dm(t)\right) \tag{3.4}$$

Using the continuous form of the arithmetic-geometric mean inequality [9, p. 64] we obtain for $r \neq \rho$ and $p > 0$

$$\begin{aligned} \left|f(re^{i\theta})\right|^p &= \frac{\rho^p}{\left|re^{i\theta} - \rho\right|^p \left|1 - \rho re^{i\theta}\right|^p} \exp\left(\int_{-\pi}^{\pi} \log\left|1 - re^{i(\theta - t)}\right|^{2p} dm(t)\right) \\ &\leqslant \frac{\rho^p}{\left|re^{i\theta} - \rho\right|^p \left|1 - \rho re^{i\theta}\right|^p} \left(\int_{-\pi}^{\pi} \left|1 - re^{i(\theta - t)}\right|^{2p} dm(t)\right). \end{aligned} \tag{3.5}$$

From (3.5) we thus have

$$\begin{aligned} \int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta &\leqslant \int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} \frac{\rho^p \left|1 - re^{i(\theta - t)}\right|^{2p}}{\left|re^{i\theta} - \rho\right|^p \left|1 - \rho re^{i\theta}\right|^p} dm(t)\right] d\theta \\ &= \int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} \frac{\rho^p \left|1 - re^{i(\theta - t)}\right|^{2p}}{\left|re^{i\theta} - \rho\right|^p \left|1 - \rho re^{i\theta}\right|^p} d\theta\right] dm(t) \\ &\leqslant \int_{-\pi}^{\pi} \left[\max_{-\pi \leqslant t \leqslant \pi} \int_{-\pi}^{\pi} \frac{\rho^p \left|1 - re^{i(\theta - t)}\right|^{2p}}{\left|re^{i\theta} - \rho\right|^p \left|1 - \rho re^{i\theta}\right|^p} d\theta\right] dm(t) \\ &= \rho^p \max_{-\pi \leqslant t \leqslant \pi} \int_{-\pi}^{\pi} \frac{\left|1 - re^{i(\theta - t)}\right|^{2p}}{\left|re^{i\theta} - \rho\right|^p \left|1 - \rho re^{i\theta}\right|^p} d\theta. \end{aligned} \tag{3.6}$$

Let

$$I(t) = \int_{-\pi}^{\pi} \frac{\left|1 - re^{i(\theta - t)}\right|^{2p}}{\left|re^{i\theta} - \rho\right|^p \left|1 - \rho re^{i\theta}\right|^p} d\theta. \tag{3.7}$$

According to (3.6), the theorem will be proven if it can be shown that

$$I(t) \leqslant \int_{-\pi}^{\pi} \frac{\left|1 + re^{i\theta}\right|^{2p}}{\left|re^{i\theta} - \rho\right|^p \left|1 - \rho re^{i\theta}\right|^p} d\theta \tag{3.8}$$

for $-\pi \leqslant t \leqslant \pi$. However, Clunie and Duren [2] have proven the following: Given a nonnegative measurable function $F(x)$ on $[-a, a]$, let $F^*(x)$ denote its symmetrically decreasing rearrangement as defined in [4, p. 278]. If $F(x)$,

$G(x)$ and $H(x)$ are nonnegative integrable functions on the interval $[-a, a]$, then

$$\int_{-a}^{a} F(x)G(x)H(x)\,dx \leqslant \int_{-a}^{a} F^*(x)G^*(x)H^*(x)\,dx. \tag{3.9}$$

Since for any t the rearrangement of $|1 - re^{i(\theta-t)}|^{2p}$ is $|1 + re^{i\theta}|^{2p}$ and since $|\rho - re^{i\theta}|^{-p}$ and $|1 - \rho re^{i\theta}|^{-p}$ are equal to their rearrangements, an application of (3.9) to (3.7) gives (3.8) and hence the theorem.

COROLLARY. *Let $f(z)$ be in $\Lambda^*(\rho)$. If*

$$f(z) = \sum_{n=1}^{\infty} B_{-n}z^{-n} + \sum_{n=0}^{\infty} B_n z^n \quad \textit{for } \rho < |z| < 1,$$

then

$$\sum_{n=1}^{\infty} |B_{-n}|^2 + \sum_{n=0}^{\infty} |B_n|^2 \leqslant \frac{2\rho^2}{(1-\rho)^2}\left[\frac{1+\rho}{1-\rho} + 2\right]. \tag{3.10}$$

PROOF. From Theorem 5 we have that for $\rho < r < 1$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\,d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2\,d\theta \tag{3.11}$$

where $F(z) = -\rho(1+z)^2/(z-\rho)(1-\rho z)$. If we let $F(z) = \sum_{n=1}^{\infty} A_{-n}z^{-n} + \sum_{n=0}^{\infty} A_n z^n$ for $\rho < |z| < 1$ then (3.11) implies that

$$\sum_{n=1}^{\infty} |B_{-n}|^2 r^{-2n} + \sum_{n=0}^{\infty} |B_n|^2 r^{2n} \leqslant \sum_{n=1}^{\infty} |A_{-n}|^2 r^{-2n} + \sum_{n=0}^{\infty} |A_n|^2 r^{2n}. \tag{3.12}$$

But it is easily seen that $A_{-n} = A_n = -\rho^n(1+\rho)/(1-\rho)$ for $n = 1, 2, 3, \ldots$ and $A_0 = -2\rho/(1-\rho)$. Using these values in (3.12) we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} |B_{-n}|^2 r^{-2n} + \sum_{n=0}^{\infty} |B_n|^2 r^{2n} \\ &\quad \leqslant \left(\frac{1+\rho}{1-\rho}\right)^2 \left[\frac{\rho^2}{r^2-\rho^2} + \frac{\rho^2 r^2}{1-\rho^2 r^2}\right] + \frac{4\rho^2}{(1-\rho)^2}. \end{aligned} \tag{3.13}$$

Letting r approach 1 in (3.13) gives (3.10).

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