

WEAKLY STARLIKE MEROMORPHIC UNIVALENT FUNCTIONS. II

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ABSTRACT. A function $f(z)$ is said to be meromorphic weakly starlike if it has the form $f(z) = -\rho z F(Z)/(z - \rho)(1 - \rho z)$ for $0 < \rho < 1$ where $F(z)$ is a member of Σ^* , the class of meromorphic, normalized starlike univalent functions. The coefficients of the power series expansion in $|z| < \rho$ of a meromorphic weakly starlike function are studied. The integral means of such functions are also discussed.

1. Introduction. Let $f(z)$ be meromorphic in the unit disk $\Delta = \{z \mid |z| < 1\}$ with a simple pole at $z = \rho$, $0 < \rho < 1$, and otherwise regular in Δ . $f(z)$ is said to be a member of $\Lambda(\rho)$ if and only if $f(0) = 1$ and there is a number ρ_1 , $\rho < \rho_1 < 1$, such that with $z = re^{i\theta}$

$$(1.1) \quad \operatorname{Re} [zf'(z)/f(z)] < 0$$

and

$$(1.2) \quad \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] d\theta = -1$$

for $\rho_1 < r < 1$. Also let $\Lambda^*(\rho)$, $0 < \rho < 1$, be the class of functions $f(z)$ which have the representation

$$(1.3) \quad f(z) = -\rho z g(z)/(z - \rho)(1 - \rho z)$$

where $g(z) = z^{-1} + a_0 + a_1 z + \dots$ is a member of Σ^* , the class of meromorphic, normalized, starlike univalent functions [1].

The classes $\Lambda(\rho)$ and $\Lambda^*(\rho)$ have recently been studied by Libera and the author [8]. Functions in $\Lambda^*(\rho)$ are called weakly starlike meromorphic univalent functions since they are reciprocals of weakly starlike regular univalent functions introduced by Hummel [5], [6]. It was pointed out in [8] that functions in $\Lambda^*(\rho)$ are univalent and for each ρ , $\Lambda(\rho)$ is a subset of $\Lambda^*(\rho)$. Furthermore if $\rho > 1/2$, $\Lambda(\rho)$ is a proper subset of $\Lambda^*(\rho)$, but if

$$0 < \rho < (3 - 2\sqrt{2})^{1/2} \approx .4,$$

then $\Lambda(\rho) = \Lambda^*(\rho)$.

If $f(z)$ is in $\Lambda^*(\rho)$ then it has a Taylor series expansion in $|z| < \rho$ of the

Received by the editors April 12, 1976.

AMS (MOS) subject classifications (1970). Primary 30A32, 30A34.

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form $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$. §2 of this paper concerns itself with the coefficients a_n of this expansion. In particular, if $f(z)$ is real for real values of z , we obtain precise upper and lower bounds on a_n for all n . §3 deals with integral means of a function in $\Lambda^*(\rho)$.

2. Coefficient bounds. Goodman [3] studied the class TM of functions $f(z)$ meromorphic in Δ with Taylor series in a neighborhood of the origin of the form $f(z) = z + \sum_{n=1}^{\infty} b_n z^n$ and which except at the poles satisfy the condition $\text{Im } f(z)\text{Im } z \geq 0$. Such functions are said to be meromorphic and typically real. Letting $TM(\rho)$ denote the subclass of TM of functions for which the poles ρ_j satisfy $|\rho_j| \geq \rho, j = 1, 2, 3, \dots$, Goodman proved [3] that for all n

$$(2.1) \quad |b_n| \leq (1 - \rho^{2n})/\rho^{n-1}(1 - \rho^2).$$

If we assume that $f(z)$ in $\Lambda^*(\rho)$ is real for real values of z and has the Taylor series expansion $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ for $|z| < \rho$, then since $f(z)$ is univalent it follows that the function $(f(z) - 1)/a_1$ is a member of $TM(\rho)$. We thus obtain from (2.1) the inequality,

$$(2.2) \quad |a_n| \leq \frac{1 - \rho^{2n}}{\rho^{n-1}(1 - \rho^2)} |a_1|$$

for all n . In [8], it was proven that $|a_1| \leq (1 + \rho)^2/\rho$. Combining this with (2.2) we obtain

$$(2.3) \quad |a_n| \leq \left(\frac{1 + \rho}{1 - \rho} \right) \left(\frac{1 - \rho^{2n}}{\rho^n} \right)$$

for all n . The interesting thing here however, is that under the assumption that $f(z)$ is real for real values of z , the coefficients a_n are necessarily positive and we can obtain precise lower bounds.

THEOREM 1. *If $f(z)$ is in $\Lambda^*(\rho)$ and real for real values of z and if $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ for $|z| < \rho$, then for all n*

$$(2.4) \quad \left(\frac{1 - \rho}{1 + \rho} \right) \left(\frac{1 - \rho^{2n}}{\rho^n} \right) \leq a_n \leq \left(\frac{1 + \rho}{1 - \rho} \right) \left(\frac{1 - \rho^{2n}}{\rho^n} \right).$$

The inequalities are sharp for each n . Equality is attained on the right side of (2.4) for each n by the function $f(z) = -\rho(1 + z)^2/(z - \rho)(1 - \rho z)$ and on the left side of (2.4) for each n by the function $f(z) = -\rho(1 - z)^2/(z - \rho)(1 - \rho z)$.

REMARK. It was proven in [8] that the function

$$f(z) = -\rho(1 - z)^2/(z - \rho)(1 - \rho z)$$

is actually in $\Lambda(\rho)$ for all ρ whereas the function

$$f(z) = -\rho(1 + z)^2/(z - \rho)(1 - \rho z)$$

is in $\Lambda(\rho)$ for $\rho \leq 1/2$ but is in $\Lambda^*(\rho) \setminus \Lambda(\rho)$ for $\rho > 1/2$. Thus the left-hand

side of (2.4) is sharp in $\Lambda(\rho)$ for all ρ and the right side is sharp in $\Lambda(\rho)$ at least for $\rho < 1/2$.

PROOF OF THEOREM 1. It follows from (1.3) that the function

$$(2.5) \quad P(z) = zf'(z)/f(z) + \rho/(z - \rho) - \rho z/(1 - \rho z).$$

has negative real part in Δ . Let

$$(2.6) \quad \frac{zf'(z)}{f(z)} = \sum_{n=1}^{\infty} b_n z^n$$

for $|z| < \rho$. Then from (2.5) we obtain

$$P(z) = -1 + \sum_{n=1}^{\infty} \left(b_n - \rho^n - \frac{1}{\rho^n} \right).$$

Since $P(z)$ has negative real part in Δ it follows that for each n

$$(2.7) \quad |b_n - \rho^n - 1/\rho^n| < 2.$$

Since by assumption $f(z)$ is real for real values of z , the coefficients b_n are real and we have from (2.7)

$$(2.8) \quad (1 - \rho^n)^2/\rho^n \leq b_n \leq (1 + \rho^n)^2/\rho^n$$

for all n . From (2.6) we have that

$$(2.9) \quad zf'(z) = \left(\sum_{n=1}^{\infty} b_n z^n \right) f(z)$$

for $|z| < \rho$. Comparing the coefficients of both sides of (2.9) we obtain

$$(2.10) \quad na_n = b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_{n-1} b_1$$

for all n . Thus in particular $a_1 = b_1 \geq (1 - \rho)^2/\rho$ which is the left side of (2.4) for $n = 1$. Next suppose that $a_k \geq ((1 - \rho)/(1 + \rho))((1 - \rho)^{2k}/\rho^k)$ for $k = 1, 2, \dots, (n - 1)$. Then from (2.4) we obtain

$$\begin{aligned} na_n &\geq \frac{(1 - \rho^n)^2}{\rho^n} + \sum_{k=1}^{n-1} \frac{(1 - \rho)}{(1 + \rho)} \frac{(1 - \rho^{2k})(1 - \rho^{n-k})^2}{\rho^n} \\ &= \frac{(1 - \rho)}{\rho^n(1 + \rho)} \left[(1 - \rho^2) \left(\sum_{k=0}^{n-1} \rho^k \right)^2 + \sum_{k=1}^{n-1} (1 - \rho^{2k})(1 - \rho^{n-k})^2 \right] \\ &= \frac{(1 - \rho)}{\rho^n(1 + \rho)} \left[(1 - \rho^2)(1 + 2\rho + 3\rho^2 + \dots + n\rho^{n-1} + (n - 1)\rho^n \right. \\ &\quad \left. + (n - 2)\rho^{n+1} + \dots + \rho^{2n-2}) + (n - 1)(1 - \rho^{2n}) \right. \\ &\quad \left. - 2 \sum_{k=1}^{n-1} \rho^{n-k} + 2 \sum_{k=1}^{n-1} \rho^{n+k} \right] \\ &= n((1 - \rho)/(1 + \rho))((1 - \rho^{2n})/\rho^n). \end{aligned}$$

Thus the left side of (2.4) follows for all n . Due to the remarks preceding Theorem 1, the right side of (2.4) now follows. In conclusion, we remark that the right side of (2.4) cannot be obtained for all n by making use of (2.10) and the right side of (2.8).

COROLLARY 1. *Let $f(z)$ be in $\Lambda^*(\rho)$. If $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ for $|z| < \rho$ and if a_n is real for $1 \leq n \leq N$, then*

$$a_n \geq ((1 - \rho) / (1 + \rho))((1 - \rho^{2n}) / \rho^n)$$

for $1 \leq n \leq N$. The inequalities are sharp.

PROOF. If $zf'(z)/f(z) = P(z) = \sum_{n=1}^{\infty} b_n z^n$ for $|z| < \rho$, then the assumptions of the corollary imply that b_n is real for $1 \leq n \leq N$. Thus (2.8) holds for $1 \leq n \leq N$. With this observation the proof proceeds as in Theorem 1.

COROLLARY 2. *Let $f(z)$ be in $\Lambda^*(\rho)$. If $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ for $|z| < \rho$ and if a_n is real for $1 \leq n \leq N$ then*

$$\operatorname{Re} a_{N+1} \geq ((1 - \rho) / (1 + \rho))((1 - \rho^{2N+1}) / \rho^{N+1}).$$

PROOF. As in the proof of Corollary 1 the coefficients b_n are real for $1 \leq n \leq N$. Thus taking real parts of both sides of (2.10) when $n = N + 1$, we can follow the argument in the proof of Theorem 1 to obtain the desired inequality.

In [3] Goodman conjectured that if $f(z)$ is meromorphic and univalent in Δ with a simple pole at $z = \rho$, $0 < \rho < 1$, and if $f(z) = z + \sum_{n=2}^{\infty} b_n z^n$ for $|z| < \rho$ then $|b_n| \leq (1 - \rho^{2n}) / \rho^{n-1}(1 - \rho^2)$. Subsequently Jenkins [7] proved that if the Bieberbach conjecture is valid up to index N , then Goodman's conjecture is also valid up to index N . Using the fact that if $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ is a member of $\Lambda^*(\rho)$, $(f(z) - 1) / a_1$ is univalent in Δ with a simple pole at $z = \rho$ and using the fact that $|a_1| \leq (1 + \rho^2) / \rho$ [8], it follows that

$$(2.11) \quad |a_n| \leq ((1 + \rho) / (1 - \rho))((1 - \rho^{2n}) / \rho^n), \quad n \leq 6.$$

With the assumption that $f(z)$ in $\Lambda^*(\rho)$ is real on the real axis we have obtained positive lower bounds on a_n for all n . It is natural then to ask whether there is in general a positive lower bound on $|a_n|$ for all $f(z)$ in $\Lambda^*(\rho)$.

3. Integral means. In this section we prove that the function

$$F(z) = -\rho(1 + z)^2 / (z - \rho)(1 - \rho z)$$

has maximal L^p means in the class $\Lambda^*(\rho)$.

THEOREM 5. *Let $f(z)$ be in $\Lambda^*(\rho)$ and $F(z) = -\rho(1 + z)^2 / (z - \rho)(1 - \rho z)$ then for any positive real number p and $r \neq \rho$,*

$$(3.1) \quad \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |F(re^{i\theta})|^p d\theta.$$

PROOF. Since $f(z)$ is a member of $\Lambda^*(\rho)$ we can write

$$(3.2) \quad f(z) = -\rho z G(z) / (z - \rho)(1 - \rho z)$$

where $G(z)$ is in Σ^* . Since $G(z)$ is in Σ^* there exists a nondecreasing function $m(t)$ on $[-\pi, \pi]$ with $\int_{-\pi}^{\pi} dm(t) = 1$ such that

$$(3.3) \quad G(z) = \frac{1}{z} \exp\left(2 \int_{-\pi}^{\pi} \log(1 - e^{-it}z) dm(t)\right).$$

Thus from (3.2) and (3.3) we have

$$(3.4) \quad f(z) = \frac{-\rho}{(z - \rho)(1 - \rho z)} \exp\left(2 \int_{-\pi}^{\pi} \log(1 - e^{-it}z) dm(t)\right)$$

Using the continuous form of the arithmetic-geometric mean inequality [9, p. 64] we obtain for $r \neq \rho$ and $p > 0$

$$(3.5) \quad \begin{aligned} |f(re^{i\theta})|^p &= \frac{\rho^p}{|re^{i\theta} - \rho|^p |1 - \rho re^{i\theta}|^p} \exp\left(\int_{-\pi}^{\pi} \log|1 - re^{i(\theta-t)}|^{2p} dm(t)\right) \\ &\leq \frac{\rho^p}{|re^{i\theta} - \rho|^p |1 - \rho re^{i\theta}|^p} \left(\int_{-\pi}^{\pi} |1 - re^{i(\theta-t)}|^{2p} dm(t)\right). \end{aligned}$$

From (3.5) we thus have

$$(3.6) \quad \begin{aligned} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta &\leq \int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} \frac{\rho^p |1 - re^{i(\theta-t)}|^{2p}}{|re^{i\theta} - \rho|^p |1 - \rho re^{i\theta}|^p} dm(t) \right] d\theta \\ &= \int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} \frac{\rho^p |1 - re^{i(\theta-t)}|^{2p}}{|re^{i\theta} - \rho|^p |1 - \rho re^{i\theta}|^p} d\theta \right] dm(t) \\ &\leq \int_{-\pi}^{\pi} \left[\max_{-\pi < t < \pi} \int_{-\pi}^{\pi} \frac{\rho^p |1 - re^{i(\theta-t)}|^{2p}}{|re^{i\theta} - \rho|^p |1 - \rho re^{i\theta}|^p} d\theta \right] dm(t) \\ &= \rho^p \max_{-\pi < t < \pi} \int_{-\pi}^{\pi} \frac{|1 - re^{i(\theta-t)}|^{2p}}{|re^{i\theta} - \rho|^p |1 - \rho re^{i\theta}|^p} d\theta. \end{aligned}$$

Let

$$(3.7) \quad I(t) = \int_{-\pi}^{\pi} \frac{|1 - re^{i(\theta-t)}|^{2p}}{|re^{i\theta} - \rho|^p |1 - \rho re^{i\theta}|^p} d\theta.$$

According to (3.6), the theorem will be proven if it can be shown that

$$(3.8) \quad I(t) \leq \int_{-\pi}^{\pi} \frac{|1 + re^{i\theta}|^{2p}}{|re^{i\theta} - \rho|^p |1 - \rho re^{i\theta}|^p} d\theta$$

for $-\pi < t < \pi$. However, Clunie and Duren [2] have proven the following: Given a nonnegative measurable function $F(x)$ on $[-a, a]$, let $F^*(x)$ denote its symmetrically decreasing rearrangement as defined in [4, p. 278]. If $F(x)$,

$G(x)$ and $H(x)$ are nonnegative integrable functions on the interval $[-a, a]$, then

$$(3.9) \quad \int_{-a}^a F(x)G(x)H(x) dx \leq \int_{-a}^a F^*(x)G^*(x)H^*(x) dx.$$

Since for any t the rearrangement of $|1 - re^{i(\theta-t)}|^{2p}$ is $|1 + re^{i\theta}|^{2p}$ and since $|\rho - re^{i\theta}|^{-p}$ and $|1 - \rho re^{i\theta}|^{-p}$ are equal to their rearrangements, an application of (3.9) to (3.7) gives (3.8) and hence the theorem.

COROLLARY. Let $f(z)$ be in $\Lambda^*(\rho)$. If

$$f(z) = \sum_{n=1}^{\infty} B_{-n}z^{-n} + \sum_{n=0}^{\infty} B_n z^n \quad \text{for } \rho < |z| < 1,$$

then

$$(3.10) \quad \sum_{n=1}^{\infty} |B_{-n}|^2 + \sum_{n=0}^{\infty} |B_n|^2 \leq \frac{2\rho^2}{(1-\rho)^2} \left[\frac{1+\rho}{1-\rho} + 2 \right].$$

PROOF. From Theorem 5 we have that for $\rho < r < 1$

$$(3.11) \quad \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^2 d\theta$$

where $F(z) = -\rho(1+z)^2/(z-\rho)(1-\rho z)$. If we let $F(z) = \sum_{n=1}^{\infty} A_{-n}z^{-n} + \sum_{n=0}^{\infty} A_n z^n$ for $\rho < |z| < 1$ then (3.11) implies that

$$(3.12) \quad \sum_{n=1}^{\infty} |B_{-n}|^2 r^{-2n} + \sum_{n=0}^{\infty} |B_n|^2 r^{2n} < \sum_{n=1}^{\infty} |A_{-n}|^2 r^{-2n} + \sum_{n=0}^{\infty} |A_n|^2 r^{2n}.$$

But it is easily seen that $A_{-n} = A_n = -\rho^n(1+\rho)/(1-\rho)$ for $n = 1, 2, 3, \dots$ and $A_0 = -2\rho/(1-\rho)$. Using these values in (3.12) we obtain

$$(3.13) \quad \sum_{n=1}^{\infty} |B_{-n}|^2 r^{-2n} + \sum_{n=0}^{\infty} |B_n|^2 r^{2n} < \left(\frac{1+\rho}{1-\rho} \right)^2 \left[\frac{\rho^2}{r^2 - \rho^2} + \frac{\rho^2 r^2}{1 - \rho^2 r^2} \right] + \frac{4\rho^2}{(1-\rho)^2}.$$

Letting r approach 1 in (3.13) gives (3.10).

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