PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 62, Number 1, January 1977

AN EXAMPLE OF AN INFINITE LIE GROUP[1]

DOMINGOS PISANELLI

ABSTRACT. We study the complex l.c.s. X of germs of holomorphic mappings around the origin of C^n, with values in C^n, vanishing at the origin. We show that X is isomorphic to $M(n, C) \times H_2$, where $M(n, C)$ is the set of complex matrices $n \times n$ and H_2 is the vector topological subspace of X of germs with vanishing jacobian matrix at the origin. We study the subset Ω of invertible germs of X. We show that Ω is open, connected and that $\pi_1(\Omega) = \mathbf{Z}$. We define in Ω a topological and a Lie group structure. We determine its infinitesimal transformation, the differential equation of its law of composition and a fundamental bound of its right side.

This work is a part of a larger research on infinite Lie groups, which started with a summary of results in $[\mathbf{P}_1]$.

In a subsequent paper we shall study the covering group of Ω.

NOTATIONS.

$$|t| = \sup_{1 \leqslant i \leqslant n} |t_i| \quad \text{when } t \in C^n.$$

$$|A| = \sup_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |a_{ij}| \quad \text{when } A = (a_{ij}) \in M(n, C),$$

the set of complex matrices $n \times n$. I the identity of $M(n, C)$.

(I) Let X be the vector space over C of germs of holomorphic mappings, defined around the origin of C^n, with values in C^n, vanishing at the origin.

We define in X two inductive limit topologies of families of subspaces indexed by $s > 0$:

$$X_s = \left\{ x \in X \,|\, \|x\|_s = \sup_{|t| < s} |J(x)(t)| < +\infty \right\}$$

and

$$Y_s = \left\{ x \in X \,|\, |x|_s = \sup_{|t| < s} |x(t)| < +\infty \right\}$$

Received by the editors February 27, 1975.

AMS (MOS) subject classifications (1970). Primary 22E65; Secondary 46F15.

Key words and phrases. Locally convex space, germ of holomorphic mapping, inductive limit topology, Silva space, direct topological sum, topological group, Lie group, local Lie group, *LF*-analyticity, Lie algebra.

[1] This work has been done in part under "Fundação de Amparo e Pesquisa do Estado de São Paulo" grant.

where $J(x)(t)$ is the matrix whose elements are $(\partial x_i/\partial t_j)(t)$ $(1 \leqslant i,j \leqslant n)$. We have $X = \bigcup_{0<s} X_s = \bigcup_{0<s} Y_s$.

$|x|_s$ is obviously a norm on Y_s and $\|x\|_s$ on X_s, for $\|x^1 + x^2\|_s \leqslant \|x^1\|_s + \|x^2\|_s$, $\|\alpha x\|_s = |\alpha|\,\|x\|_s$, and

$$0 = \|x\|_s \Rightarrow (\partial x_i/\partial t_j)(t) = 0 \quad \forall |t| < s, \qquad 1 \leqslant i,j \leqslant n.$$

$x(t) = \text{constant} = x(0) = 0$ $\forall |t| < s$. Whenever $x, x^1, x^2 \in Y_s$, $\alpha \in C$. Cauchy inequality gives

$$\left|\frac{\partial x_i}{\partial t_j}(t)\right| \leqslant \frac{1}{s - s'} \sup_{|t|<s} |x_i(t)|$$

where $1 \leqslant i,j \leqslant n$, and $|t| < s' < s$. Then

(1) $$\|x\|_{s'} \leqslant n|x|_s/(s - s') \quad \forall s' < s.$$

The Lagrange inequality gives also

$$|x(t)| \leqslant |t| \sup_{\alpha \in |0,t|} \sup_{|h|\leqslant 1} |x'(\alpha)h|$$
$$= |t| \sup_{\alpha \in |0,t|} \sup_{|h|\leqslant 1} |J(x)(\alpha)h|;$$

then

(2) $$|x|_s \leqslant s\|h\|_s.$$

(1) and (2) show that $Y_s \hookrightarrow X_{s'}$ $\forall s' < s$, $X_s \hookrightarrow Y_s$ $\forall s$ and lim inj Y_s = lim inj X_s, i.e. *the inductive limit topology of* X_s *and* Y_s *coincide*.

Y_s is obviously a Banach space. X_s is also a Banach space. For let $(x_n)_{n\geqslant 1}$ be a Cauchy sequence in X_s. $(x_n)_{n\geqslant 1}$ is also a Cauchy sequence in Y_s. $x_n \to x \in Y_s$ in Y_s and $J(x_n)(t) \to J(x)(t)$ for each t such that $|t| < s$. But for $\varepsilon > 0$ we have $n_0(\varepsilon)$ such that

$$|J(x_n)(t) - J(x_m)(t)| < \varepsilon \Leftarrow n, m \geqslant n_0(\varepsilon), \qquad |t| < s,$$

and

$$|J(x)(t) - J(x_m)(t)| \leqslant \varepsilon \Leftarrow m \geqslant n_0(\varepsilon), \qquad |t| < s.$$

Then $x \in X_s$ and $x_m \to x$ in X_s.

$\bigcup_{0<s} Y_s$ is a Silva space [S_2], for a ball of Y_s centered at the origin is relatively compact in $Y_{s'}$ $(\forall s' < s)$ by Montel's theorem.

$\bigcup_{0<s} X_s$ *is a Silva space* for $X_s \hookrightarrow Y_s \hookrightarrow X_{s''}$ when $s'' < s' < s$.

(II) X is the direct algebraic sum of $H_1 = \{\dot{At} \in X | A \in M(n, C)\}$ and $H_2 = \{x \in X | J(x)(0) = 0\}$.

Here $\dot{At}$ is the germ of the mapping $t \in C^n \to At \in C^n$.

For the proof it is enough to apply the Taylor series of x.

X is the direct topological sum of H_1 and H_2: $X_s \ni x \to J(x)(0)t \in Y_s$ is continuous $\forall s$, for $|J(x)(0)t|_s \leqslant s\|x\|_s$; then $X \ni x \to J(x)(0)t \in H_1$ is continuous, in the inductive limit topologies of X_s and Y_s induced on H_1.

H_1 is a subspace of finite dimension n^2, so it is isomorphic to $C^{n^2} \sim M(n, C)$.

As a consequence

$$X = H_1 \oplus H_2 \sim H_1 \times H_2 \sim M(n, C) \times H_2.$$

(III) Let Ω be the set of elements $x \in X$ whose jacobian determinant at the origin is different from zero.

Ω is homeomorphic to $\mathrm{Gl}(n, C) \times H_2$.

As a consequence Ω *is open and connected*, for $\mathrm{Gl}(n, C)$ is open and connected in C^{n^2}. $\pi_1(\Omega)$, *the first homotopy group of* Ω, *is* $\mathbf{Z}$ for $\mathrm{Gl}(n, C)$ is homeomorphic to $U(n) \times R^{n^2}$, where $U(n)$ is the set of unitary matrices [**C**, p. 16] and $\pi_1(U(n)) = \mathbf{Z}$ [**H**, p. 93].

For $x, y \in \Omega$, let $x \circ y \in \Omega$ be the element defined by $x \circ y(t) = x(y(t))$. Endowed with this operation Ω *is a group*; the identity e is the germ of the mapping $e(t) = t$ and the inverse x^{-1} of the element $x \in \Omega$ is the germ of the inverse mapping of a representative of x.

Definition. Let X_1 and Y_1 be l.c.s. Hausdorff and sequentially complete, let Ω_1 be open in X_1 and let f be a mapping defined in Ω_1 with values in Y_1. By definition *f is LF-analytic* when $f \circ g$ is analytic for any g analytic function of a complex variable with values in Ω_1.

Remark 1. Let g be analytic in $D \subset C$ with values in $X = \bigcup_{0<s} X_s$.

(a) If we restrict g to a ball $D_r(\alpha_0) \subset \overline{D}_r(\alpha_0) \subset D$, there exists $s > 0$ such that g is analytic with values in X_s.

(b) $g(\alpha)(t)$ is analytic in $D_r(\alpha_0) \times (|t| < s)$ and $g(\alpha, 0) = 0$.

For there exists $s > 0$ such that $g(D_r(\alpha_0))$ is contained in X_s, moreover $g(D_r(\alpha_0))$ is bounded in X_s [$\mathbf{S_2}$]. Then there exists $M > 0$ such that $\|g(\alpha)\|_s \leqslant M$ ($\forall \alpha \in D_r(\alpha_0)$). Cauchy inequality,

$$\|g^{(n)}(\alpha_0)/n!\|_s \leqslant M/r^n \qquad (n \geqslant 0),$$

shows that

$$g(\alpha) = \sum_{n \geqslant 0} \frac{g^{(n)}(\alpha_0)}{n!}(\alpha - \alpha_0)^n$$

converges in X_s when $|\alpha - \alpha_0| < r$. This proves (a). (b) is immediate.

Remark 2. Let g: $D_r(\alpha_0) \times (|t| < s) \to C$ be analytic and $g(\alpha, 0) = 0$; then $\alpha \in D_r(\alpha_0) \to g(\alpha) \in \bigcup_{0<s} X_s$ $(g(\alpha)(t) = g(\alpha, t))$ is analytic.

For the proof use Cauchy integral formula.

Theorem. *The mappings*

(A) $$x, y \in \Omega \to x \circ y \in \Omega,$$

(B) $$x \in \Omega \to x^{-1} \in \Omega$$

are LF-analytic.

In the proof of (A) it is enough to show that:

(3) $$x \in \Omega \to x \circ y \in \Omega \quad \text{is } LF\text{-analytic } (\forall y \in \Omega).$$

(4) $$y \in \Omega \to x \circ y \in \Omega \quad \text{is } LF\text{-analytic } (\forall x \in \Omega).$$

Let $x(\alpha, t)$ be analytic in $D_r(\alpha_0) \times (|t| < s)$ with values in C^n (consider Remark 1), and $y(t)$ analytic in $|t| < s'$ valued in $|t| < s$. $x(\alpha, y(t))$ is analytic in $D_r(\alpha_0) \times (|t| < s')$. By Remark 2 $x(\alpha) \circ y$ is analytic.

Let $x(t)$ be analytic in $|t| < s$ with values in C^n and $y(\alpha, t)$ analytic in $D_r(\alpha_0) \times (|t| < s')$ with values in $|t| < s$ (consider Remark 1). $x(y(\alpha, t))$ is analytic in $D_r(\alpha_0) \times (|t| < s')$. Again by Remark 2, $x \circ y(\alpha)$ is analytic.

An easy generalization of Hartog's theorem [$\mathbf{P_2}$] shows that (3) and (4) imply (A).

Let us prove that $x \in \Omega \to x^{-1} \in \Omega$ is LF-analytic.

Let $x(\alpha, t)$ be analytic in a neighbourhood of $(\alpha_0, 0) \in C \times C^n$, $x(\alpha, 0) = 0$ and $\det J(x(\alpha_0, 0)) \neq 0$ (see Remark 1).

By the implicit function theorem there exists $g(\alpha, t)$ analytic in $D_r(\alpha_0) \times (|t| < s)$ such that $g(\alpha, x(\alpha, t)) = t$. Then by Remark 2, $g(\alpha)$ is analytic and $g(\alpha) \circ x(\alpha) = e$, i.e. $(x(\alpha))^{-1}$ is analytic.

A theorem of J. S. Silva [$\mathbf{S_1}$] gives us the continuity of LF-analytic mappings defined in an open set of a Silva space [$\mathbf{S_1}$]. The product of Silva spaces is a Silva space (use [$\mathbf{S_2}$, Theorem 1, p. 399]),

$$(x,y) \in \Omega \times \Omega \to x \circ y \in \Omega,$$

$$x \in \Omega \to x^{-1} \in \Omega$$

are continuous.

COROLLARY. *Ω is a topological group.*

(IV) **Differential equation of the law of composition of** Ω. Let $\phi(x,y) = x \circ y$. The associative property gives $\phi(\phi(x,y), z) = \phi(x, \phi(y,z))$, $\forall x, y, z \in \Omega$.

Differentiating both sides with respect to z at the point $z = e$ and with increment h we have [$\mathbf{P_2}$]:

$$\phi'_z(\phi(x,y), e)h = \phi'_z(x,y)\phi'_z(y,e)h,$$

$$L(y)h = \phi'_z(y,e)h = \{(d/d\alpha)y \circ (e + \alpha h)\}_{\alpha=0}$$

(5) $$= \left(\left\{\frac{d}{d\alpha} y_j \circ (e + \alpha h)\right\}_{\alpha=0}\right) = \left(\left\{\sum_{i=1}^{n} \left(\frac{\partial y_j}{\partial t_i}\right)_{e+\alpha h} h_i\right\}_{\alpha=0}\right)$$

$$= \left(\sum_{i=1}^{n} \frac{\partial y_j}{\partial t_i} h_i\right) = J(y)h.$$

From (5) we deduce the following differential equation for the law of composition of Ω:

$$\phi'_y(x,y)h = J(\phi)J(y)^{-1}h, \qquad \phi(x,e) = x. \tag{6}$$

To determine the *Lie algebra of the Lie group* Ω we shall compute

$$\begin{aligned} |h,k| &= L'(e)kh - L'(e)hk \\ &= \{(d/d\alpha)J(e+\alpha k)h\}_{\alpha=0} - \{(d/d\alpha)J(e+\alpha h)k\}_{\alpha=0} \\ &= J(k)h - J(h)k. \end{aligned}$$

Bound of the right side of (6).

Let $\|x-e\|_s < r < 1$. Then $||J(x)(t)|^{-1}| \leqslant 1/(1-r)$ when $|t| < s$.

$$(J(x)(t))^{-1} = (I - J(e-x)(t))^{-1} = \sum_{n\geqslant 0}(J(e-x)(t))^n,$$

for

$$\sum_{n\geqslant 0}|J(e-x)(t)|^n \leqslant \sum_{n\geqslant 0} r^n = \frac{1}{1-r} \quad \text{when } |t| < s.$$

Inequalities (1) and (2) imply that

$$\|J(\phi)J(y)^{-1}h\|_{s'} \leqslant \frac{n}{s-s'}|J(\phi)J(y)^{-1}h|_s \leqslant \frac{n}{s-s'}\|\phi\|_s\frac{1}{1-r}s\|h\|_s;$$

then

$$\|J(\phi)J(y)^{-1}h\|_s \leqslant n\frac{1+r}{1-r}\frac{s}{s-s'}\|h\|_s,$$

i.e.

$$\|L(\phi)L(y)^{-1}h\|_{s'} \leqslant cs\|h\|_s/(s-s') \tag{7}$$

(c independent from s, s') if $\|\phi - e\|_s < r$, $\|y-e\|_s < r$, $h \in X_s$, $s' < s$.

(7) is the appropriate inequality in order to construct a local Lie group in a Banach scale, from an infinitesimal transformation $L(y)h$ [P_1].

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Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brasil