

AN EXAMPLE OF AN INFINITE LIE GROUP¹

DOMINGOS PISANELLI

ABSTRACT. We study the complex l.c.s. X of germs of holomorphic mappings around the origin of C^n , with values in C^n , vanishing at the origin. We show that X is isomorphic to $M(n, C) \times H_2$, where $M(n, C)$ is the set of complex matrices $n \times n$ and H_2 is the vector topological subspace of X of germs with vanishing jacobian matrix at the origin. We study the subset Ω of invertible germs of X . We show that Ω is open, connected and that $\pi_1(\Omega) = \mathbb{Z}$. We define in Ω a topological and a Lie group structure. We determine its infinitesimal transformation, the differential equation of its law of composition and a fundamental bound of its right side.

This work is a part of a larger research on infinite Lie groups, which started with a summary of results in [P₁].

In a subsequent paper we shall study the covering group of Ω .

NOTATIONS.

$$|t| = \sup_{1 \leq i \leq n} |t_i| \quad \text{when } t \in C^n.$$

$$|A| = \sup_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}| \quad \text{when } A = (a_{ij}) \in M(n, C),$$

the set of complex matrices $n \times n$. I the identity of $M(n, C)$.

(I) Let X be the vector space over C of germs of holomorphic mappings, defined around the origin of C^n , with values in C^n , vanishing at the origin.

We define in X two inductive limit topologies of families of subspaces indexed by $s > 0$:

$$X_s = \left\{ x \in X \mid \|x\|_s = \sup_{|t| < s} |J(x)(t)| < +\infty \right\}$$

and

$$Y_s = \left\{ x \in X \mid |x|_s = \sup_{|t| < s} |x(t)| < +\infty \right\}$$

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where $J(x)(t)$ is the matrix whose elements are $(\partial x_i / \partial t_j)(t)$ ($1 \leq i, j \leq n$). We have $X = \bigcup_{0 < s} X_s = \bigcup_{0 < s} Y_s$.

$|x|_s$ is obviously a norm on Y_s and $\|x\|_s$ on X_s , for $\|x^1 + x^2\|_s \leq \|x^1\|_s + \|x^2\|_s$, $\|\alpha x\|_s = |\alpha| \|x\|_s$, and

$$0 = \|x\|_s \Rightarrow (\partial x_i / \partial t_j)(t) = 0 \quad \forall |t| < s, \quad 1 \leq i, j \leq n.$$

$x(t) = \text{constant} = x(0) = 0 \quad \forall |t| < s$. Whenever $x, x^1, x^2 \in Y_s, \alpha \in C$. Cauchy inequality gives

$$\left| \frac{\partial x_i}{\partial t_j}(t) \right| \leq \frac{1}{s - s'} \sup_{|t| < s} |x_i(t)|$$

where $1 \leq i, j \leq n$, and $|t| < s' < s$. Then

$$(1) \quad \|x\|_{s'} \leq n|x|_s / (s - s') \quad \forall s' < s.$$

The Lagrange inequality gives also

$$\begin{aligned} |x(t)| &\leq |t| \sup_{\alpha \in [0, t]} \sup_{|h| < 1} |x'(\alpha)h| \\ &= |t| \sup_{\alpha \in [0, t]} \sup_{|h| < 1} |J(x)(\alpha)h|; \end{aligned}$$

then

$$(2) \quad |x|_s \leq s \|h\|_s.$$

(1) and (2) show that $Y_s \hookrightarrow X_{s'} \quad \forall s' < s, X_s \hookrightarrow Y_{s'} \quad \forall s'$ and $\lim \text{inj } Y_s = \lim \text{inj } X_s$, i.e. the inductive limit topology of X_s and Y_s coincide.

Y_s is obviously a Banach space. X_s is also a Banach space. For let $(x_n)_{n \geq 1}$ be a Cauchy sequence in X_s . $(x_n)_{n \geq 1}$ is also a Cauchy sequence in Y_s . $x_n \rightarrow x \in Y_s$ in Y_s and $J(x_n)(t) \rightarrow J(x)(t)$ for each t such that $|t| < s$. But for $\epsilon > 0$ we have $n_0(\epsilon)$ such that

$$|J(x_n)(t) - J(x_m)(t)| < \epsilon \iff n, m \geq n_0(\epsilon), \quad |t| < s,$$

and

$$|J(x)(t) - J(x_m)(t)| \leq \epsilon \iff m \geq n_0(\epsilon), \quad |t| < s.$$

Then $x \in X_s$ and $x_m \rightarrow x$ in X_s .

$\bigcup_{0 < s} Y_s$ is a Silva space $[S_2]$, for a ball of Y_s centered at the origin is relatively compact in $Y_{s'}$ ($\forall s' < s$) by Montel's theorem.

$\bigcup_{0 < s} X_s$ is a Silva space for $X_s \hookrightarrow Y_s \hookrightarrow X_{s'}$ when $s'' < s' < s$.

(II) X is the direct algebraic sum of $H_1 = \{At \in X | A \in M(n, C)\}$ and $H_2 = \{x \in X | J(x)(0) = 0\}$.

Here At is the germ of the mapping $t \in C^n \rightarrow At \in C^n$.

For the proof it is enough to apply the Taylor series of x .

X is the direct topological sum of H_1 and H_2 ; $X_s \ni x \rightarrow J(x)(0)t \in Y_s$ is continuous $\forall s$, for $\|J(x)(0)t\|_s \leq s\|x\|_s$; then $X \ni x \rightarrow J(x)(0)t \in H_1$ is continuous, in the inductive limit topologies of X_s and Y_s induced on H_1 .

H_1 is a subspace of finite dimension n^2 , so it is isomorphic to $C^{n^2} \sim M(n, C)$.

As a consequence

$$X = H_1 \oplus H_2 \sim H_1 \times H_2 \sim M(n, C) \times H_2.$$

(III) Let Ω be the set of elements $x \in X$ whose jacobian determinant at the origin is different from zero.

Ω is homeomorphic to $Gl(n, C) \times H_2$.

As a consequence Ω is open and connected, for $Gl(n, C)$ is open and connected in C^{n^2} . $\pi_1(\Omega)$, the first homotopy group of Ω , is \mathbf{Z} for $Gl(n, C)$ is homeomorphic to $U(n) \times R^{n^2}$, where $U(n)$ is the set of unitary matrices [C, p. 16] and $\pi_1(U(n)) = \mathbf{Z}$ [H, p. 93].

For $x, y \in \Omega$, let $x \circ y \in \Omega$ be the element defined by $x \circ y(t) = x(y(t))$. Endowed with this operation Ω is a group; the identity e is the germ of the mapping $e(t) = t$ and the inverse x^{-1} of the element $x \in \Omega$ is the germ of the inverse mapping of a representative of x .

DEFINITION. Let X_1 and Y_1 be l.c.s. Hausdorff and sequentially complete, let Ω_1 be open in X_1 and let f be a mapping defined in Ω_1 with values in Y_1 . By definition f is LF-analytic when $f \circ g$ is analytic for any g analytic function of a complex variable with values in Ω_1 .

REMARK 1. Let g be analytic in $D \subset C$ with values in $X = \cup_{0 < s} X_s$.

(a) If we restrict g to a ball $D_r(\alpha_0) \subset \bar{D}_r(\alpha_0) \subset D$, there exists $s > 0$ such that g is analytic with values in X_s .

(b) $g(\alpha)(t)$ is analytic in $D_r(\alpha_0) \times (|t| < s)$ and $g(\alpha, 0) = 0$.

For there exists $s > 0$ such that $g(D_r(\alpha_0))$ is contained in X_s , moreover $g(D_r(\alpha_0))$ is bounded in X_s [S_2]. Then there exists $M > 0$ such that $\|g(\alpha)\|_s \leq M (\forall \alpha \in D_r(\alpha_0))$. Cauchy inequality,

$$\|g^{(n)}(\alpha_0)/n!\|_s \leq M/r^n \quad (n \geq 0),$$

shows that

$$g(\alpha) = \sum_{n \geq 0} \frac{g^{(n)}(\alpha_0)}{n!} (\alpha - \alpha_0)^n$$

converges in X_s when $|\alpha - \alpha_0| < r$. This proves (a). (b) is immediate.

REMARK 2. Let $g: D_r(\alpha_0) \times (|t| < s) \rightarrow C$ be analytic and $g(\alpha, 0) = 0$; then $\alpha \in D_r(\alpha_0) \rightarrow g(\alpha) \in \cup_{0 < s} X_s$ ($g(\alpha)(t) = g(\alpha, t)$) is analytic.

For the proof use Cauchy integral formula.

THEOREM. *The mappings*

(A) $x, y \in \Omega \rightarrow x \circ y \in \Omega,$

(B) $x \in \Omega \rightarrow x^{-1} \in \Omega$

are *LF-analytic*.

In the proof of (A) it is enough to show that:

(3) $x \in \Omega \rightarrow x \circ y \in \Omega$ is *LF-analytic* ($\forall y \in \Omega$).

(4) $y \in \Omega \rightarrow x \circ y \in \Omega$ is *LF-analytic* ($\forall x \in \Omega$).

Let $x(\alpha, t)$ be analytic in $D_r(\alpha_0) \times (|t| < s)$ with values in C^n (consider Remark 1), and $y(t)$ analytic in $|t| < s'$ valued in $|t| < s$. $x(\alpha, y(t))$ is analytic in $D_r(\alpha_0) \times (|t| < s')$. By Remark 2 $x(\alpha) \circ y$ is analytic.

Let $x(t)$ be analytic in $|t| < s$ with values in C^n and $y(\alpha, t)$ analytic in $D_r(\alpha_0) \times (|t| < s')$ with values in $|t| < s$ (consider Remark 1). $x(y(\alpha, t))$ is analytic in $D_r(\alpha_0) \times (|t| < s')$. Again by Remark 2, $x \circ y(\alpha)$ is analytic.

An easy generalization of Hartog's theorem [P₂] shows that (3) and (4) imply (A).

Let us prove that $x \in \Omega \rightarrow x^{-1} \in \Omega$ is *LF-analytic*.

Let $x(\alpha, t)$ be analytic in a neighbourhood of $(\alpha_0, 0) \in C \times C^n$, $x(\alpha, 0) = 0$ and $\det J(x(\alpha_0, 0)) \neq 0$ (see Remark 1).

By the implicit function theorem there exists $g(\alpha, t)$ analytic in $D_r(\alpha_0) \times (|t| < s)$ such that $g(\alpha, x(\alpha, t)) = t$. Then by Remark 2, $g(\alpha)$ is analytic and $g(\alpha) \circ x(\alpha) = e$, i.e. $(x(\alpha))^{-1}$ is analytic.

A theorem of J. S. Silva [S₁] gives us the continuity of *LF-analytic* mappings defined in an open set of a Silva space [S₁]. The product of Silva spaces is a Silva space (use [S₂, Theorem 1, p. 399]),

$$(x, y) \in \Omega \times \Omega \rightarrow x \circ y \in \Omega,$$

$$x \in \Omega \rightarrow x^{-1} \in \Omega$$

are continuous.

COROLLARY. Ω is a topological group.

(IV) **Differential equation of the law of composition of Ω .** Let $\phi(x, y) = x \circ y$. The associative property gives $\phi(\phi(x, y), z) = \phi(x, \phi(y, z)), \forall x, y, z \in \Omega$.

Differentiating both sides with respect to z at the point $z = e$ and with increment h we have [P₂]:

$$\phi'_z(\phi(x, y), e)h = \phi'_z(x, y)\phi'_z(y, e)h,$$

$$L(y)h = \phi'_z(y, e)h = \{(d/d\alpha)y \circ (e + \alpha h)\}_{\alpha=0}$$

$$(5) \quad = \left(\left\{ \frac{d}{d\alpha} y^j \circ (e + \alpha h) \right\}_{\alpha=0} \right) = \left(\left\{ \sum_{i=1}^n \left(\frac{\partial y_j}{\partial t_i} \right)_{e+\alpha h} h_i \right\}_{\alpha=0} \right)$$

$$= \left(\sum_{i=1}^n \frac{\partial y_j}{\partial t_i} h_i \right) = J(y)h.$$

From (5) we deduce the following differential equation for the law of composition of Ω :

$$(6) \quad \phi'_j(x, y)h = J(\phi)J(y)^{-1}h, \quad \phi(x, e) = x.$$

To determine the Lie algebra of the Lie group Ω we shall compute

$$\begin{aligned} |h, k| &= L'(e)kh - L'(e)hk \\ &= \{(d/d\alpha)J(e + \alpha k)h\}_{\alpha=0} - \{(d/d\alpha)J(e + \alpha h)k\}_{\alpha=0} \\ &= J(k)h - J(h)k. \end{aligned}$$

Bound of the right side of (6).

Let $\|x - e\|_s < r < 1$. Then $\|J(x)(t)^{-1}\| \leq 1/(1-r)$ when $|t| < s$.

$$(J(x)(t))^{-1} = (I - J(e-x)(t))^{-1} = \sum_{n \geq 0} (J(e-x)(t))^n,$$

for

$$\sum_{n \geq 0} |J(e-x)(t)|^n \leq \sum_{n \geq 0} r^n = \frac{1}{1-r} \quad \text{when } |t| < s.$$

Inequalities (1) and (2) imply that

$$\|J(\phi)J(y)^{-1}h\|_{s'} \leq \frac{n}{s-s'} |J(\phi)J(y)^{-1}h|_s \leq \frac{n}{s-s'} \|\phi\|_s \frac{1}{1-r} s \|h\|_s;$$

then

$$\|J(\phi)J(y)^{-1}h\|_s \leq n \frac{1+r}{1-r} \frac{s}{s-s'} \|h\|_s,$$

i.e.

$$(7) \quad \|L(\phi)L(y)^{-1}h\|_{s'} \leq cs \|h\|_{s'}/(s-s')$$

(c independent from s, s') if $\|\phi - e\|_s < r, \|y - e\|_s < r, h \in X_s, s' < s$.

(7) is the appropriate inequality in order to construct a local Lie group in a Banach scale, from an infinitesimal transformation $L(y)h$ [\mathbf{P}_1].

BIBLIOGRAPHY

[\mathbf{P}_1] D. Pisanelli, *Théorèmes d'Ovcyannicov, Frobenius d'inversion et groupes de Lie locaux dans une échelle d'espaces de Banach*, C. R. Acad. Sci. Paris **277** (1973), 943-946.

[\mathbf{P}_2] ———, *Applications analytiques en dimension infinie*, Bull. Sci. Math. **96** (1972), 181-191.

[\mathbf{S}_1] José Sebastião e Silva, *Sui fondamenti della teoria dei funzionali analitici*, Portugal. Math. **12** (1953), 1-47. MR **14**, 656.

[\mathbf{S}_2] ———, *Su certe classi di spazi localmente convessi importanti per le applicazioni*, Rend. Mat. e Appl. (5) **14** (1955), 388-410. MR **16**, 1122.

[\mathbf{C}] C. Chevalley, *Theory of Lie groups*, Princeton Univ. Press, Princeton, N.J., 1946. MR **7**, 412.

[\mathbf{H}] D. Husemoller, *Fibre bundles*, McGraw-Hill, New York, 1966. MR **37** #4821.

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, SÃO PAULO, BRASIL