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AN $L_{\omega_1 \omega}$ COMPLETE AND CONSISTENT THEORY WITHOUT MODELS

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ABSTRACT. We simplify the example of Ryll-Nardzewski of an $L_{\omega_1 \omega}$ theory with the above properties. We include other facts on the relationship of logic to 01-laws and on the $L_{\omega_1 \omega}$ definability of some sets which are meager but of full measure.

As mentioned by Scott [1] the first example of a theory T of this kind was invented by Ryll-Nardzewski in 1962 (unpublished). To make this within a finite similarity type, the proof of Ryll-Nardzewski had a nontrivial inductive argument. Here we give a simpler example and proof.

Let Z be the set of all integers and C be the space of all subsets of Z with the usual topology and measure.

Let T_t be the set of all sentences α of $L_{\omega_1 \omega}$ with equality, of the appropriate similarity type, such that the set $S_\alpha = \{P \in C: \langle Z, <, P\rangle \models \alpha\}$ is comeager. Let T_m be defined in the same way except that now S_α is to be of measure 1.

THEOREM. *Both T_t and T_m are*
(i) *countably satisfiable,*
(ii) *complete,*
(iii) *without models.*

PROOF. (i) follows from the theorem of Baire and the countable additivity of measure.

(ii) For every α the set S_α is Borel, invariant under shift, and hence, by the 01-laws, is meager or comeager and of measure 0 or of measure 1. Thus, for every α, either α or $\neg\alpha$ is in T_t and the same is true for T_m.

(iii) Suppose, to the contrary, that M is a model of T_t or T_m. $\langle Z, <\rangle$ is describable up to isomorphism by a sentence of $L_{\omega_1 \omega}$ which belongs to T_t and T_m, namely the sentence which says that $<$ is a linear ordering without first or last element, and that for every x, y the set $\{z: x < z < y\}$ is finite. Hence M is isomorphic to a structure of the form $\langle Z, <, P\rangle$, where $P \subseteq Z$. Thus M is countable and, hence, is describable up to isomorphism by its Scott sentence α_M (see [1]). Then all the models of α_M which are of the form $\langle Z, <, P\rangle$ are isomorphic by shifts. Hence S_{α_M} is countable, meager and of measure 0. Thus $\neg\alpha_M \in T_t$ and $\neg\alpha_M \in T_m$, and M cannot be a model of T_t nor of T_m.

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REMARK 1. $T_t \neq T_m$.

PROOF. By [1] and [2] for every Borel set $A \subseteq C$ invariant under shifts there exists a sentence α of $L_{\omega_1\omega}$ such that $A = \{P \in C: \langle Z, <, P\rangle \models \alpha\}$. Hence there exists a sentence α_0 of $L_{\omega_1\omega}$ such that

$$\langle Z, <, P\rangle \models \alpha_0 \quad \text{iff} \lim_{n\to\infty} \frac{\operatorname{card}(P \cap [-n,n])}{2n} = \frac{1}{2}.$$

(A direct construction of α_0 is also easy.) Therefore by the strong law of large numbers $\alpha_0 \in T_m$. On the other hand, it is well known and readily seen from a Banach-Mazur game that the set

$$\{P \in C: \forall m \exists n > m[\operatorname{card}(P \cap [-n,n]) < n/2]\}$$

is comeager. Hence $\alpha_0 \notin T_t$.

REMARK 2. Let two sequences $r_0, r_1, \ldots$ and $p_0, p_1, \ldots$ of positive integers be given. We consider structures of the type $\langle Z, R_i, P_j\rangle_{i,j<\omega}$, where R_i are any fixed relations of ranks r_i, which are invariant under shift, i.e., such that $R_i(x_1, \ldots, x_{r_i}) \leftrightarrow R_i(x_1 + 1, \ldots, x_{r_i} + 1)$, and P_j are variable relations of fixed ranks p_j. Then for every sentence $\alpha \in L_{\omega_1\omega}$ of the appropriate language the set

$$S_\alpha = \{\langle P_j\rangle_{j<\omega}: \langle Z, R_i, P_j\rangle_{i,j<\omega} \models \alpha\}$$

is meager or comeager and of measure 0 or 1 in the appropriate space. Hence the theories T_t and T_m can be defined again and are countably satisfiable and complete, but may have models. (See also [3].)

REMARK 3. With such a generalization there may exist a *finite* sentence $\alpha_0 \in T_t$ which is not in T_m. In fact consider the structures $\langle Z, R_+, R_\times, P\rangle$, where $R_+, R_\times \subseteq Z^4$,

$$R_+(x,y,z,t) \leftrightarrow x + y - t = z,$$

$$R_\times(x,y,z,t) \leftrightarrow (x - t)(y - t) + t = z,$$

and $P \subseteq Z$. Then R_+ and $R_\times$ are invariant under shift. The construction of α_0 is similar to the construction in Remark 1, but here we use the fact that the relation $\operatorname{card}(P \cap [2t - x, x]) = y - t$ can be expressed by a finite formula $\varphi(x,y,t)$ of our language. Also $x < y$ can be expressed. Hence we can also express the relation $\operatorname{card}(P \cap [2t - x, x]) < x/2$ by some formula $\psi(x,t)$. Then the sentence $\exists t \forall y (\exists x > y)\psi(x,t)$ can be expressed and, by the 01-laws, the set of its models is comeager but of measure 0.

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