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GROWTH NEAR THE BOUNDARY IN $H^2(\mu)$ SPACES[1]

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ABSTRACT. Let $H^2(\mu)$ be the closure in $L^2(\mu)$ of the complex polynomials, where μ is a finite Borel measure supported on the closed unit disk in the complex plane. For $|z| < 1$, let $E(z) \equiv \sup|p(z)|/\|p\|$ where the supremum is over all polynomials p whose $L^2(\mu)$ norm $\|p\|$ is nonzero. An inequality is derived asymptotically relating $E(z)$ (as z tends to the unit circle) to the part of μ supported on the unit circle. The interplay between μ and the growth of functions in $H^2(\mu)$ is studied in the event that $E(z) < \infty$ for $|z| < 1$.

1. **General $H^2(\mu)$ spaces over the disk.** Let μ be a finite positive compactly supported Borel measure on the complex plane. It is well known that many interesting questions in the theory of subnormal operators can be reduced to function-theoretic questions about $H^2(\mu)$, the subspace of $L^2(\mu)$ spanned by all polynomials [1], [2], [3]. The purpose of this note is to investigate the relationship between μ and the quantity $E(z)$ defined in the abstract, under the assumption that μ is supported on the closure $\bar{D}$ of the open unit disk D. Note that $E(z) < \infty$ if and only if point evaluation at z is a bounded linear functional on polynomials with respect to the $L^2(\mu)$ norm, and in this case $E(z)$ is the norm of the evaluation functional. We consider $E(z)$ only for z in D and we will be interested in the asymptotic behavior of $E(z)$ as z tends to the boundary ∂D. In §2 we consider applications to the growth of functions in $H^2(\mu)$ when $E(z) < \infty$ for z in D.

Let α be the part of μ supported on ∂D and ν the part supported on D, so that $\mu = \alpha + \nu$. Let σ denote normalized Lebesgue measure on ∂D, $d\sigma(\theta) = d\theta/2\pi$, $0 \leqslant \theta < 2\pi$, and suppose that α_0 is the part of α which is absolutely continuous with respect to σ. Let w denote any fixed representative of $d\alpha_0/d\sigma$. We use the convention that $1/0 = \infty$ and $\infty \cdot a = \infty$ for $a > 0$.

THEOREM. *With the understanding that $z \to e^{i\theta}$ nontangentially we have*

$$\liminf_{z \to e^{i\theta}} (1 - |z|^2) E(z)^2 \geqslant 1/w(e^{i\theta}) \quad \sigma\text{-a.e.} \tag{1}$$

If in addition $\int \log w \, d\sigma > -\infty$, then $E(z) < \infty$ for all z in D and

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(2) $$\lim_{z \to e^{i\theta}} (1 - |z|^2) E(z)^2 = 1/w(e^{i\theta}) \quad \sigma\text{-}a.e.$$

To facilitate the proof we introduce the notation

$$P(z, u) = (1 - |z|^2)/|1 - \bar{z}u|^2, \qquad \bar{z}u \neq 1.$$

The Poisson integral of an f in $L^1(\sigma)$ is then

(3) $$f(z) = \int P(z, e^{ix}) f(e^{ix})\, d\sigma(x), \qquad |z| < 1.$$

Analogously, if β is a finite positive measure supported on $\bar{D}$ we define

$$\hat{\beta}(z) = \int P(z, u)\, d\beta(u), \qquad |z| < 1.$$

Note that $\hat{\beta}$ is not, in general, harmonic since $P(z, u)$ is not harmonic in z for fixed u in D.

LEMMA 1. $\lim_{z \to e^{i\theta}} \hat{\nu}(z) = 0$ *for σ-almost every $e^{i\theta}$, where the limit is taken nontangentially.*

PROOF. For $k = 0, 1, 2, \ldots$ let R_k be defined by

$$R_k(e^{ix}) = \int P(z, e^{ix}) |z|^{2k}\, d\nu(z).$$

Since $\nu(\partial D) = 0$ an application of the Fubini-Tonelli theorems shows that R_k is in $L^1(\sigma)$ and indeed, that

(4) $$\int_{\partial D} f R_k\, d\sigma = \int_D f(z) |z|^{2k}\, d\nu(z)$$

for every f continuous on $\bar{D}$ and harmonic in D (to see that (4) holds, substitute the representation (3) for $f(z)$ into the right side and use Fubini's theorem). Thus $R_k\, d\sigma$ is the "sweep" of $|z|^{2k}\, d\nu(z)$ to the boundary. This concrete representation for balayage was brought to our attention by S. Clary's thesis [4].

Now let F_k denote the Poisson integral of R_k:

$$F_k(z) = \int P(z, e^{ix}) R_k(e^{ix})\, d\sigma(x), \qquad |z| < 1.$$

Fatou's theorem [6, p. 34] implies that for σ-almost every $e^{i\theta}$, $F_k(z)$ tends to $R_k(e^{i\theta})$ as $z \to e^{i\theta}$ nontangentially. On applying (4) to the continuous harmonic function

$$f(u) = \operatorname{Re}[(1 + \bar{z}u)/(1 - \bar{z}u)] \quad (\text{for } z \text{ fixed in } D)$$

we have

$$\int \operatorname{Re}\left(\frac{1 + \bar{z}u}{1 - \bar{z}u} \right) |u|^{2k}\, d\nu(u) = \int \operatorname{Re}\left(\frac{1 + \bar{z}e^{ix}}{1 - \bar{z}e^{ix}} \right) R_k(e^{ix})\, d\sigma(x)$$
$$= \int P(z, e^{ix}) R_k(e^{ix})\, d\sigma(x) = F_k(z).$$

We can now compute

$$\hat{\nu}(z) = \int P(z,u)\,d\nu(u) = \int \frac{1-|z|^2}{1-|z|^2|u|^2}\operatorname{Re}\left(\frac{1+\bar{z}u}{1-\bar{z}u}\right)d\nu(u)$$

$$(5) \qquad = \left(1-|z|^2\right)\sum_{j=0}^{\infty}|z|^{2j}\int \operatorname{Re}\left(\frac{1+\bar{z}u}{1-\bar{z}u}\right)|u|^{2j}\,d\nu(u)$$

$$= (1-|z|^2)\sum_{j=0}^{\infty}|z|^{2j}F_j(z).$$

Fix $e^{i\theta}$ such that $R_0(e^{i\theta}) < \infty$ and $F_j(z) \to R_j(e^{i\theta})$ for $j = 0, 1, 2, \ldots$ as $z \to e^{i\theta}$ nontangentially. Clearly this holds for σ-almost every $e^{i\theta}$. The condition $R_0(e^{i\theta}) < \infty$ tells us (by definition) that $P(z, e^{i\theta})$ (as a function of z) belongs to $L^1(\nu)$. Since ν is supported on D, $P(z, e^{i\theta})|z|^{2k}\downarrow 0$ ν-a.e. as $k \to \infty$, and we conclude from the Lebesgue dominated convergence theorem that $R_k(e^{i\theta}) \to 0$ as $k \to \infty$. Further, since $R_{j+1}(e^{i\theta}) \leqslant R_j(e^{i\theta})$ for all j, we see that $F_n(z) \leqslant F_k(z)$ whenever $n \geqslant k$.

Suppose that we are given $\varepsilon > 0$ and let $z \to e^{i\theta}$ nontangentially. It suffices to show that

$$(6) \qquad \limsup_{z\to e^{i\theta}} \hat{\nu}(z) \leqslant 2\varepsilon.$$

Fix k large enough so that $R_k(e^{i\theta}) < \varepsilon$. Then if z is sufficiently close to $e^{i\theta}$ we have, for any $n \geqslant k$,

$$F_n(z) \leqslant F_k(z) < R_k(e^{i\theta}) + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon.$$

Combining this with (5) gives

$$\hat{\nu}(z) \leqslant (1-|z|^2)\left[\sum_{j=0}^{k-1}|z|^{2j}F_j(z) + \sum_{j=k}^{\infty}2\varepsilon|z|^{2j}\right]$$

$$= (1-|z|^2)\sum_{j=0}^{k-1}|z|^{2j}F_j(z) + 2\varepsilon|z|^{2k}.$$

Since $F_j(z) \to R_j(e^{i\theta}) < \infty$ as $z \to e^{i\theta}$ for all j, the first term on the right tends to zero and (6) is established. This completes the proof.

PROOF OF THE THEOREM. For fixed z in D the function $(1-\bar{z}u)^{-1}$ is a uniform limit of polynomials in $\overline{D}$ and thus belongs to $H^2(\mu)$. Therefore,

$$1/|1-\bar{z}u|^2 \leqslant E(u)^2\int\left(1/|1-\bar{z}s|^2\right)d\mu(s)$$

for all u in D. Setting $u = z$ and multiplying by $(1-|z|^2)$ gives

$$1/(1-|z|^2) \leqslant E(z)^2\int\frac{1-|z|^2}{|1-\bar{z}s|^2}\,d\mu(s)$$

or equivalently,

$$(7) \qquad 1/\hat{\mu}(z) \leqslant (1-|z|^2)E(z)^2.$$

With our previous notation we have $\hat{\mu}(z) = \hat{\alpha}(z) + \hat{\nu}(z)$. Lemma 1 tells us that for σ-almost every $e^{i\theta}$, $\hat{\nu}(z) \to 0$ as $z \to e^{i\theta}$; Fatou's theorem, on the other hand, implies that $\hat{\alpha}(z) \to w(e^{i\theta})$ σ-a.e. Thus $\hat{\mu}(z)$ tends to $w(e^{i\theta})$ as $z \to e^{i\theta}$ nontangentially, at least σ-a.e., and (1) follows from taking the lim inf of both sides of (7) as z tends to $e^{i\theta}$.

Now assume that log w is σ-integrable and select an outer function g in the Hardy space $H^2(\sigma)$ with $w = |g|^2$ σ-a.e. [6, p. 53]. Then for any z in D and any polynomial p

$$|p(z)|^2 \leqslant |g(z)|^{-2}(1 - |z|^2)^{-1} \int |p|^2 w \, d\sigma. \tag{8}$$

This follows from [5, p. 48].

Since (8) holds for all polynomials p and $\int |p|^2 w \, d\sigma \leqslant \int |p|^2 \, d\mu$ we have

$$E(z)^2 \leqslant |g(z)|^{-2}(1 - |z|^2)^{-1}, \qquad |z| < 1.$$

Inasmuch as $\lim_{z \to e^{i\theta}} |g(z)|^2 = w(e^{i\theta})$ σ-a.e. we find that

$$\limsup_{z \to e^{i\theta}} (1 - |z|^2) E(z)^2 \leqslant 1/w(e^{i\theta}) \quad \sigma\text{-a.e.},$$

which in combination with (1) proves (2). This completes the proof.

2. **Functional growth when $E(z)$ is finite on D.** We make the standing assumption in this section that $E(z) < \infty$ for all z in D. Then for each z the evaluation functional $p \to p(z)$ on polynomials has a unique bounded extension to all of $H^2(\mu)$ giving rise to an unambiguous determination of $f(z)$ for all f in $H^2(\mu)$. For each z there exists a unique element K_z of $H^2(\mu)$ (the kernel function) such that

$$f(z) = \int f \overline{K_z} \, d\mu, \qquad f \text{ in } H^2(\mu).$$

Clearly $\|K_z\| = E(z)$.

LEMMA 2. *Let m be a finite Borel measure on ∂D such that the inequality* (1) (*which holds σ-a.e. by the theorem*) *holds m-a.e. Suppose that Ω is a positive continuous function on* (0, 1]. *If*

$$(1 - r^2) E(re^{i\theta})^2 \leqslant M \cdot (1 - r^2)/\Omega(1 - r^2) \tag{9}$$

for every f in $H^2(\mu)$, then there exists a positive constant M such that

$$\frac{1}{w(e^{i\theta})} \leqslant M \cdot \left(\liminf_{t \to 0} \frac{t}{\Omega(t)} \right) \quad m\text{-a.e.}$$

PROOF. Assume that (9) holds for every f in $H^2(\mu)$. We then have a well-defined linear map L: $H^2(\mu) \to L^\infty(r\, dr\, dm(\theta))$ given by $(Lf)(re^{i\theta}) = \Omega(1 - r^2)^{1/2} f(re^{i\theta})$. L is readily seen to be closed, so it is bounded by the closed graph theorem. Thus there exists $M > 0$ such that for every polynomial p,

$$\Omega(1 - r^2) |p(re^{i\theta})|^2 \leqslant M \|p\|^2$$

for all r in $[0, 1)$ and all $e^{i\theta}$ in the closed support of m; here we are using the continuity of Ω and p. It follows that

$$m\text{-}\operatorname*{ess\,sup}_{e^{i\theta}} \left(\sup_{0 \leqslant r < 1} \Omega(1 - r^2) |f(re^{i\theta})|^2 \right) < \infty \tag{10}$$

for all $e^{i\theta}$ outside of some m-null set and all r in $[0, 1)$. We may now take the lim inf as $r \to 1$ in (10) and use the hypothesis on m to complete the proof.

COROLLARY 1. *Let Ω be a positive continuous function on* $(0, 1]$ *with* $\liminf_{t\to 0} t/\Omega(t) = 0$. *Then for σ-almost every $e^{i\theta}$ in ∂D there exists an f in $H^2(\mu)$ (depending on $e^{i\theta}$) with*

$$\sup_{0<r<1} \Omega(1 - r^2)|f(re^{i\theta})|^2 = \infty.$$

PROOF. By the Theorem we can select a set $G \subset \partial D$ with $\sigma(G) = 0$ and so that inequality (1) holds for *every* $e^{i\theta}$ in $\partial D \setminus G$. Suppose that $e^{i\theta}$ is a point in $\partial D \setminus G$ such that the supremum in the statement is finite for every f in $H^2(\mu)$. By taking m to be a unit point mass at $e^{i\theta}$ we may conclude from Lemma 2 and the hypothesis on Ω that $1/w(e^{i\theta}) = 0$. As this can only happen for $e^{i\theta}$ in a set of σ-measure zero, the proof is complete.

COROLLARY 2. *Let $F \subset \partial D$ be a set of positive σ-measure such that the σ-essential infimum of w on F is zero. Then there exists an f in $H^2(\mu)$ with*

$$\sup_{e^{i\theta} \text{ in } F;\ 0<r<1} (1 - r^2)|f(re^{i\theta})|^2 = \infty.$$

PROOF. We apply Lemma 2, taking m to be the restriction of σ to F and $\Omega(t) = t$. If the conclusion of the corollary fails, the Theorem and Lemma 2 together imply the existence of $M > 0$ such that $1/w(e^{i\theta}) \leqslant M$ σ-a.e. on F, which contradicts the hypothesis on F.

COROLLARY 3. *Let $Q = \{e^{i\theta}: w(e^{i\theta}) = 0\}$. Then for σ-almost every $e^{i\theta}$ in Q there exists an f in $H^2(\mu)$ (depending on $e^{i\theta}$) with*

$$\sup_{0<r<1} (1 - r^2)|f(re^{i\theta})|^2 = \infty.$$

PROOF. Select the set G as in the proof of Corollary 1. If there is an $e^{i\theta}$ in $Q \setminus G$ such that the supremum in the statement is finite for every f in $H^2(\mu)$, we may apply Lemma 2 with $\Omega(t) = t$ and m a unit point mass at $e^{i\theta}$ to conclude that $w(e^{i\theta}) > 0$, a contradiction. The proof is complete.

3. **Conclusion.** If $\int \log w \, d\sigma = -\infty$ and $\mu = w \, d\sigma$, then $E(z) \equiv \infty$ on D [5, p. 48] so that (2) does not hold in general. However we know of no counterexample to (2) with the property that $E(z) < \infty$ for all z in D. When $\int \log w \, d\sigma = -\infty$ our simple proof of (2) of course fails and any method of estimating $E(z)$ from above for the purpose of proving (2) must necessarily take account of the possibly complex interplay between w and ν, the part of μ supported on D. A proof of (2) in any great generality might require an answer to the difficult question: when, in terms of μ, is $E(z)$ finite? Of course, when $w = 0$ σ-a.e. (1) implies (2) trivially. A more interesting instance of (2) occurs when μ is the measure associated with the discrete Cesàro operator [7]. Here $w(e^{i\theta})$ and $E(z) = \|K_z\|$ are explicitly computable, $\int \log w \, d\sigma = -\infty$ and (2) is easily verified [7, §1].

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