

GROWTH NEAR THE BOUNDARY IN $H^2(\mu)$ SPACES¹

THOMAS KRIETE AND TAVAN TRENT

ABSTRACT. Let $H^2(\mu)$ be the closure in $L^2(\mu)$ of the complex polynomials, where μ is a finite Borel measure supported on the closed unit disk in the complex plane. For $|z| < 1$, let $E(z) \equiv \sup |p(z)|/\|p\|$ where the supremum is over all polynomials p whose $L^2(\mu)$ norm $\|p\|$ is nonzero. An inequality is derived asymptotically relating $E(z)$ (as z tends to the unit circle) to the part of μ supported on the unit circle. The interplay between μ and the growth of functions in $H^2(\mu)$ is studied in the event that $E(z) < \infty$ for $|z| < 1$.

1. General $H^2(\mu)$ spaces over the disk. Let μ be a finite positive compactly supported Borel measure on the complex plane. It is well known that many interesting questions in the theory of subnormal operators can be reduced to function-theoretic questions about $H^2(\mu)$, the subspace of $L^2(\mu)$ spanned by all polynomials [1], [2], [3]. The purpose of this note is to investigate the relationship between μ and the quantity $E(z)$ defined in the abstract, under the assumption that μ is supported on the closure \bar{D} of the open unit disk D . Note that $E(z) < \infty$ if and only if point evaluation at z is a bounded linear functional on polynomials with respect to the $L^2(\mu)$ norm, and in this case $E(z)$ is the norm of the evaluation functional. We consider $E(z)$ only for z in D and we will be interested in the asymptotic behavior of $E(z)$ as z tends to the boundary ∂D . In §2 we consider applications to the growth of functions in $H^2(\mu)$ when $E(z) < \infty$ for z in D .

Let α be the part of μ supported on ∂D and ν the part supported on D , so that $\mu = \alpha + \nu$. Let σ denote normalized Lebesgue measure on ∂D , $d\sigma(\theta) = d\theta/2\pi$, $0 \leq \theta < 2\pi$, and suppose that α_0 is the part of α which is absolutely continuous with respect to σ . Let w denote any fixed representative of $d\alpha_0/d\sigma$. We use the convention that $1/0 = \infty$ and $\infty \cdot a = \infty$ for $a > 0$.

THEOREM. *With the understanding that $z \rightarrow e^{i\theta}$ nontangentially we have*

$$(1) \quad \liminf_{z \rightarrow e^{i\theta}} (1 - |z|^2)E(z)^2 \geq 1/w(e^{i\theta}) \quad \sigma\text{-a.e.}$$

If in addition $\int \log w \, d\sigma > -\infty$, then $E(z) < \infty$ for all z in D and

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$$(2) \quad \lim_{z \rightarrow e^{i\theta}} (1 - |z|^2)E(z)^2 = 1/w(e^{i\theta}) \quad \sigma\text{-a.e.}$$

To facilitate the proof we introduce the notation

$$P(z, u) = (1 - |z|^2)/|1 - \bar{z}u|^2, \quad \bar{z}u \neq 1.$$

The Poisson integral of an f in $L^1(\sigma)$ is then

$$(3) \quad f(z) = \int P(z, e^{ix})f(e^{ix}) d\sigma(x), \quad |z| < 1.$$

Analogously, if β is a finite positive measure supported on \bar{D} we define

$$\hat{\beta}(z) = \int P(z, u) d\beta(u), \quad |z| < 1.$$

Note that $\hat{\beta}$ is not, in general, harmonic since $P(z, u)$ is not harmonic in z for fixed u in D .

LEMMA 1. $\lim_{z \rightarrow e^{i\theta}} \hat{\nu}(z) = 0$ for σ -almost every $e^{i\theta}$, where the limit is taken nontangentially.

PROOF. For $k = 0, 1, 2, \dots$ let R_k be defined by

$$R_k(e^{ix}) = \int P(z, e^{ix})|z|^{2k} d\nu(z).$$

Since $\nu(\partial D) = 0$ an application of the Fubini-Tonelli theorems shows that R_k is in $L^1(\sigma)$ and indeed, that

$$(4) \quad \int_{\partial D} fR_k d\sigma = \int_D f(z)|z|^{2k} d\nu(z)$$

for every f continuous on \bar{D} and harmonic in D (to see that (4) holds, substitute the representation (3) for $f(z)$ into the right side and use Fubini's theorem). Thus $R_k d\sigma$ is the "sweep" of $|z|^{2k} d\nu(z)$ to the boundary. This concrete representation for balayage was brought to our attention by S. Clary's thesis [4].

Now let F_k denote the Poisson integral of R_k :

$$F_k(z) = \int P(z, e^{ix})R_k(e^{ix}) d\sigma(x), \quad |z| < 1.$$

Fatou's theorem [6, p. 34] implies that for σ -almost every $e^{i\theta}$, $F_k(z)$ tends to $R_k(e^{i\theta})$ as $z \rightarrow e^{i\theta}$ nontangentially. On applying (4) to the continuous harmonic function

$$f(u) = \operatorname{Re}[(1 + \bar{z}u)/(1 - \bar{z}u)] \quad (\text{for } z \text{ fixed in } D)$$

we have

$$\begin{aligned} \int \operatorname{Re}\left(\frac{1 + \bar{z}u}{1 - \bar{z}u}\right)|u|^{2k} d\nu(u) &= \int \operatorname{Re}\left(\frac{1 + \bar{z}e^{ix}}{1 - \bar{z}e^{ix}}\right)R_k(e^{ix}) d\sigma(x) \\ &= \int P(z, e^{ix})R_k(e^{ix}) d\sigma(x) = F_k(z). \end{aligned}$$

We can now compute

$$\begin{aligned}
 \hat{\nu}(z) &= \int P(z, u) \, d\nu(u) = \int \frac{1 - |z|^2}{1 - |z|^2|u|^2} \operatorname{Re}\left(\frac{1 + \bar{z}u}{1 - \bar{z}u}\right) \, d\nu(u) \\
 (5) \quad &= (1 - |z|^2) \sum_{j=0}^{\infty} |z|^{2j} \int \operatorname{Re}\left(\frac{1 + \bar{z}u}{1 - \bar{z}u}\right) |u|^{2j} \, d\nu(u) \\
 &= (1 - |z|^2) \sum_{j=0}^{\infty} |z|^{2j} F_j(z).
 \end{aligned}$$

Fix $e^{i\theta}$ such that $R_0(e^{i\theta}) < \infty$ and $F_j(z) \rightarrow R_j(e^{i\theta})$ for $j = 0, 1, 2, \dots$ as $z \rightarrow e^{i\theta}$ nontangentially. Clearly this holds for σ -almost every $e^{i\theta}$. The condition $R_0(e^{i\theta}) < \infty$ tells us (by definition) that $P(z, e^{i\theta})$ (as a function of z) belongs to $L^1(\nu)$. Since ν is supported on D , $P(z, e^{i\theta})|z|^{2k} \downarrow 0$ ν -a.e. as $k \rightarrow \infty$, and we conclude from the Lebesgue dominated convergence theorem that $R_k(e^{i\theta}) \rightarrow 0$ as $k \rightarrow \infty$. Further, since $R_{j+1}(e^{i\theta}) \leq R_j(e^{i\theta})$ for all j , we see that $F_n(z) \leq F_k(z)$ whenever $n \geq k$.

Suppose that we are given $\varepsilon > 0$ and let $z \rightarrow e^{i\theta}$ nontangentially. It suffices to show that

$$(6) \quad \limsup_{z \rightarrow e^{i\theta}} \hat{\nu}(z) \leq 2\varepsilon.$$

Fix k large enough so that $R_k(e^{i\theta}) < \varepsilon$. Then if z is sufficiently close to $e^{i\theta}$ we have, for any $n \geq k$,

$$F_n(z) \leq F_k(z) \leq R_k(e^{i\theta}) + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon.$$

Combining this with (5) gives

$$\begin{aligned}
 \hat{\nu}(z) &\leq (1 - |z|^2) \left[\sum_{j=0}^{k-1} |z|^{2j} F_j(z) + \sum_{j=k}^{\infty} 2\varepsilon |z|^{2j} \right] \\
 &= (1 - |z|^2) \sum_{j=0}^{k-1} |z|^{2j} F_j(z) + 2\varepsilon |z|^{2k}.
 \end{aligned}$$

Since $F_j(z) \rightarrow R_j(e^{i\theta}) < \infty$ as $z \rightarrow e^{i\theta}$ for all j , the first term on the right tends to zero and (6) is established. This completes the proof.

PROOF OF THE THEOREM. For fixed z in D the function $(1 - \bar{z}u)^{-1}$ is a uniform limit of polynomials in \bar{D} and thus belongs to $H^2(\mu)$. Therefore,

$$1/|1 - \bar{z}u|^2 \leq E(u)^2 \int (1/|1 - \bar{z}s|^2) \, d\mu(s)$$

for all u in D . Setting $u = z$ and multiplying by $(1 - |z|^2)$ gives

$$1/(1 - |z|^2) \leq E(z)^2 \int \frac{1 - |z|^2}{|1 - \bar{z}s|^2} \, d\mu(s)$$

or equivalently,

$$(7) \quad 1/\hat{\mu}(z) \leq (1 - |z|^2)E(z)^2.$$

With our previous notation we have $\hat{\mu}(z) = \hat{\alpha}(z) + \hat{\nu}(z)$. Lemma 1 tells us that for σ -almost every $e^{i\theta}$, $\hat{\nu}(z) \rightarrow 0$ as $z \rightarrow e^{i\theta}$; Fatou's theorem, on the other

hand, implies that $\hat{\alpha}(z) \rightarrow w(e^{i\theta})$ σ -a.e. Thus $\hat{\mu}(z)$ tends to $w(e^{i\theta})$ as $z \rightarrow e^{i\theta}$ nontangentially, at least σ -a.e., and (1) follows from taking the \liminf of both sides of (7) as z tends to $e^{i\theta}$.

Now assume that $\log w$ is σ -integrable and select an outer function g in the Hardy space $H^2(\sigma)$ with $w = |g|^2$ σ -a.e. [6, p. 53]. Then for any z in D and any polynomial p

$$(8) \quad |p(z)|^2 \leq |g(z)|^{-2}(1 - |z|^2)^{-1} \int |p|^2 w \, d\sigma.$$

This follows from [5, p. 48].

Since (8) holds for all polynomials p and $\int |p|^2 w \, d\sigma \leq \int |p|^2 \, d\mu$ we have.

$$E(z)^2 \leq |g(z)|^{-2}(1 - |z|^2)^{-1}, \quad |z| < 1.$$

Inasmuch as $\lim_{z \rightarrow e^{i\theta}} |g(z)|^2 = w(e^{i\theta})$ σ -a.e. we find that

$$\limsup_{z \rightarrow e^{i\theta}} (1 - |z|^2)E(z)^2 \leq 1/w(e^{i\theta}) \quad \sigma\text{-a.e.},$$

which in combination with (1) proves (2). This completes the proof.

2. Functional growth when $E(z)$ is finite on D . We make the standing assumption in this section that $E(z) < \infty$ for all z in D . Then for each z the evaluation functional $p \rightarrow p(z)$ on polynomials has a unique bounded extension to all of $H^2(\mu)$ giving rise to an unambiguous determination of $f(z)$ for all f in $H^2(\mu)$. For each z there exists a unique element K_z of $H^2(\mu)$ (the kernel function) such that

$$f(z) = \int f \bar{K}_z \, d\mu, \quad f \text{ in } H^2(\mu).$$

Clearly $\|K_z\| = E(z)$.

LEMMA 2. *Let m be a finite Borel measure on ∂D such that the inequality (1) (which holds σ -a.e. by the theorem) holds m -a.e. Suppose that Ω is a positive continuous function on $(0, 1]$. If*

$$(9) \quad (1 - r^2)E(re^{i\theta})^2 \leq M \cdot (1 - r^2)/\Omega(1 - r^2)$$

for every f in $H^2(\mu)$, then there exists a positive constant M such that

$$\frac{1}{w(e^{i\theta})} \leq M \cdot \left(\liminf_{t \rightarrow 0} \frac{t}{\Omega(t)} \right) \quad m\text{-a.e.}$$

PROOF. Assume that (9) holds for every f in $H^2(\mu)$. We then have a well-defined linear map $L: H^2(\mu) \rightarrow L^\infty(r \, dr \, dm(\theta))$ given by $(Lf)(re^{i\theta}) = \Omega(1 - r^2)^{1/2} f(re^{i\theta})$. L is readily seen to be closed, so it is bounded by the closed graph theorem. Thus there exists $M > 0$ such that for every polynomial p ,

$$\Omega(1 - r^2)|p(re^{i\theta})|^2 \leq M\|p\|^2$$

for all r in $[0, 1)$ and all $e^{i\theta}$ in the closed support of m ; here we are using the continuity of Ω and p . It follows that

$$(10) \quad m\text{-ess sup}_{e^{i\theta}} \left(\sup_{0 < r < 1} \Omega(1 - r^2)|f(re^{i\theta})|^2 \right) < \infty$$

for all $e^{i\theta}$ outside of some m -null set and all r in $[0, 1)$. We may now take the \liminf as $r \rightarrow 1$ in (10) and use the hypothesis on m to complete the proof.

COROLLARY 1. *Let Ω be a positive continuous function on $(0, 1]$ with $\liminf_{t \rightarrow 0} t/\Omega(t) = 0$. Then for σ -almost every $e^{i\theta}$ in ∂D there exists an f in $H^2(\mu)$ (depending on $e^{i\theta}$) with*

$$\sup_{0 < r < 1} \Omega(1 - r^2) |f(re^{i\theta})|^2 = \infty.$$

PROOF. By the Theorem we can select a set $G \subset \partial D$ with $\sigma(G) = 0$ and so that inequality (1) holds for every $e^{i\theta}$ in $\partial D \setminus G$. Suppose that $e^{i\theta}$ is a point in $\partial D \setminus G$ such that the supremum in the statement is finite for every f in $H^2(\mu)$. By taking m to be a unit point mass at $e^{i\theta}$ we may conclude from Lemma 2 and the hypothesis on Ω that $1/w(e^{i\theta}) = 0$. As this can only happen for $e^{i\theta}$ in a set of σ -measure zero, the proof is complete.

COROLLARY 2. *Let $F \subset \partial D$ be a set of positive σ -measure such that the σ -essential infimum of w on F is zero. Then there exists an f in $H^2(\mu)$ with*

$$\sup_{e^{i\theta} \text{ in } F; 0 < r < 1} (1 - r^2) |f(re^{i\theta})|^2 = \infty.$$

PROOF. We apply Lemma 2, taking m to be the restriction of σ to F and $\Omega(t) = t$. If the conclusion of the corollary fails, the Theorem and Lemma 2 together imply the existence of $M > 0$ such that $1/w(e^{i\theta}) \leq M$ σ -a.e. on F , which contradicts the hypothesis on F .

COROLLARY 3. *Let $Q = \{e^{i\theta} : w(e^{i\theta}) = 0\}$. Then for σ -almost every $e^{i\theta}$ in Q there exists an f in $H^2(\mu)$ (depending on $e^{i\theta}$) with*

$$\sup_{0 < r < 1} (1 - r^2) |f(re^{i\theta})|^2 = \infty.$$

PROOF. Select the set G as in the proof of Corollary 1. If there is an $e^{i\theta}$ in $Q \setminus G$ such that the supremum in the statement is finite for every f in $H^2(\mu)$, we may apply Lemma 2 with $\Omega(t) = t$ and m a unit point mass at $e^{i\theta}$ to conclude that $w(e^{i\theta}) > 0$, a contradiction. The proof is complete.

3. Conclusion. If $\int \log w \, d\sigma = -\infty$ and $\mu = w \, d\sigma$, then $E(z) \equiv \infty$ on D [5, p. 48] so that (2) does not hold in general. However we know of no counterexample to (2) with the property that $E(z) < \infty$ for all z in D . When $\int \log w \, d\sigma = -\infty$ our simple proof of (2) of course fails and any method of estimating $E(z)$ from above for the purpose of proving (2) must necessarily take account of the possibly complex interplay between w and v , the part of μ supported on D . A proof of (2) in any great generality might require an answer to the difficult question: when, in terms of μ , is $E(z)$ finite? Of course, when $w = 0$ σ -a.e. (1) implies (2) trivially. A more interesting instance of (2) occurs when μ is the measure associated with the discrete Cesàro operator [7]. Here $w(e^{i\theta})$ and $E(z) = \|K_z\|$ are explicitly computable, $\int \log w \, d\sigma = -\infty$ and (2) is easily verified [7, §1].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22903