

## N-COMPACTNESS AND WEAK HOMOGENEITY

S. BROVERMAN<sup>1</sup>

**ABSTRACT.** A characterization of those locally compact, 0-dimensional, realcompact spaces  $X$  that are N-compact is given in terms of a density condition on  $\beta X - X$ .

**1. Introduction and preliminaries.** In [6], Mrówka introduced the concept of  $\mathbf{E}$ -compactness. Given a space  $\mathbf{E}$ , a space  $X$  is  $\mathbf{E}$ -compact if  $X$  can be embedded as a closed subspace of  $\mathbf{E}^m$  for some cardinal number  $m$ . The class of N-compact spaces has been given much attention recently where  $\mathbf{N}$  denotes the set of natural numbers. It was conjectured in [2] that the class of N-compact spaces is equal to the class of 0-dimensional realcompact spaces (a space is 0-dimensional if it has a base of clopen sets). However, Nyikos gave in [9] an example of a 0-dimensional, metric space that is realcompact but not N-compact. In [10], Walker raised the question of which realcompact, 0-dimensional, Hausdorff spaces are N-compact. In this paper we give a characterization of those locally compact, realcompact, 0-dimensional, Hausdorff spaces  $X$  that are N-compact, in terms of a density condition on  $\beta X - X$  (where  $\beta X$  denotes the Stone-Čech compactification of  $X$ ).

The following theorem will be required and is stated without proof. It can be found as Theorem 4.10 of [7].

**1.1 THEOREM.** *Let  $\mathbf{E}$  be a completely regular, Hausdorff space. Suppose  $X$  can be embedded in  $\mathbf{E}^m$  for some cardinal number  $m$ . Then  $X$  is  $\mathbf{E}$ -compact if and only if given a point  $p \in \beta X - X$  there is a map  $f: X \rightarrow \mathbf{E}$  such that  $(\beta(f))(p) \in \beta \mathbf{E} - \mathbf{E}$  (where  $\beta(f): \beta X \rightarrow \beta \mathbf{E}$  is the Stone extension of the map  $f$ ).*

All spaces are assumed to be completely regular and Hausdorff.

### 2. N-compact spaces.

**2.1 DEFINITION.** A space  $X$  is called *weakly homogeneous* if for every  $p \in \beta X - X$ , the set  $\{q \in \beta X - X: q = (\beta(f))(p) \text{ for some } f: X \rightarrow X\}$  is dense in  $\beta X - X$ .

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This condition is slightly weaker than that of almost homogeneity defined by Bellamy and Rubin [1], which requires that  $f$  be a homeomorphism of  $X$  onto  $X$ . It is not difficult to prove that  $\mathbf{N}$  and the real line  $\mathbf{R}$  are both almost homogeneous, hence, weakly homogeneous. We can go somewhat further.

**2.2 THEOREM.** *Let  $\mathbf{E} = \mathbf{I} \times \mathbf{N}$  where  $\mathbf{I}$  denotes the unit interval. Then any  $\mathbf{E}$ -compact space is weakly homogeneous.*

**PROOF.** Let  $X$  be  $\mathbf{E}$ -compact. Suppose  $p \in \beta X - X$  and  $U \subseteq \beta X - X$  is open. It is enough to show that there exists a map  $f: X \rightarrow X$  such that  $(\beta(f))(p) \in U$ .

Since  $p \in \beta X - X$  and  $X$  is  $\mathbf{E}$ -compact, by Theorem 1.1 there is a map  $k: X \rightarrow \mathbf{I} \times \mathbf{N}$  such that  $(\beta(k))(p) \in \beta\mathbf{E} - \mathbf{E}$ . Let  $h: X \rightarrow \mathbf{N}$  be defined by  $h = \pi_{\mathbf{N}} \circ k$  (where  $\pi_{\mathbf{N}}: \mathbf{I} \times \mathbf{N} \rightarrow \mathbf{N}$  is the projection map). Clearly  $(\beta(h))(p) \in \beta\mathbf{N} - \mathbf{N}$  ( $\pi_{\mathbf{N}}$  is a perfect map, hence, by [4, 10.13],  $(\beta(\pi_{\mathbf{N}}))(\beta\mathbf{E} - \mathbf{E}) \subseteq \beta\mathbf{N} - \mathbf{N}$ ).

Let  $V$  be a closed subset of  $X$  such that  $\emptyset \neq (\text{cl}_{\beta X} V) - X \subseteq U$ . Since  $V$  is not compact, and  $X$  is realcompact,  $V$  cannot be pseudocompact. Thus by [4, 1.21],  $V$  contains a  $C$ -embedded, closed copy of  $\mathbf{N}$ , say  $\{x_n: n \in \mathbf{N}\}$ .

Let  $f: X \rightarrow X$  be defined by  $f(x) = x_i$  if and only if  $h(x) = i$ . Thus  $(\beta(f))(p) \in (\text{cl}_{\beta X} V) - X \subseteq U$ . This completes the theorem.  $\square$

It is not true that every  $\mathbf{I} \times \mathbf{N}$ -compact space is almost homogeneous. A strongly 0-dimensional, realcompact space is  $\mathbf{N}$ -compact as shown by Herrlich in [5] (a space  $X$  is strongly 0-dimensional if  $\beta X$  is 0-dimensional). Let  $D$  be the discrete space of cardinality  $\aleph_1$ . Then  $D$  is realcompact and strongly 0-dimensional, hence is  $\mathbf{N}$ -compact (see [4, Theorem 12.2]). Let  $p \in \beta D - D$  such that  $p$  is not in the closure of any countable subset of  $D$  and let  $U = (\text{cl}_{\beta D} V) - D$  where  $V$  is a countably infinite subset of  $D$ . Clearly if  $h: D \rightarrow D$  is any homeomorphism, then  $(\beta(h))(p) \notin U$ . Thus  $D$  is  $\mathbf{N}$ -compact, hence  $\mathbf{I} \times \mathbf{N}$ -compact, but is not almost homogeneous.

The following theorem is required and is stated without proof. The reader may find it as Theorem 3.1 of [3].

**2.3 THEOREM.** *Let  $X$  be locally compact and realcompact. Then a subset  $D \subseteq \beta X - X$  is dense in  $\beta X - X$  if and only if  $X \cup D$  is pseudocompact.*

**2.4 THEOREM.** *Let  $X$  be locally compact. The following two conditions on  $X$  are equivalent.*

- (i)  $X$  is realcompact, 0-dimensional and weakly homogeneous,
- (ii)  $X$  is  $\mathbf{N}$ -compact.

**PROOF.** (ii) implies (i). This follows immediately from Theorem 2.2 in view of the fact that an  $\mathbf{N}$ -compact space is  $\mathbf{I} \times \mathbf{N}$ -compact, realcompact and 0-dimensional.

(i) implies (ii). Suppose  $X$  is not  $\mathbf{N}$ -compact. In particular,  $X$  is not compact, hence as  $X$  is realcompact and 0-dimensional,  $X$  is not pseudocompact and

there is an unbounded map from  $X$  to  $\mathbf{N}$ . By Theorem 1.1, there is a point  $p \in \beta X - X$  such that if  $f: X \rightarrow \mathbf{N}$  then  $(\beta(f))(p) \in \mathbf{N}$ . Let  $k: X \rightarrow X$  such that  $q = (\beta(k))(p) \in \beta X - X$ . Let  $f: X \rightarrow \mathbf{N}$ . Then

$$(\beta(f))(q) = (\beta(f)) \circ (\beta(k))(p) = (\beta(f \circ k))(p) \in \mathbf{N}$$

by the remark above.

Suppose  $f: X \rightarrow \mathbf{N}$ . Let  $A = \{q \in \beta X - X: q = (\beta(k))(p) \text{ for some } k: X \rightarrow X\}$ . Then, by what we have just shown,  $(\beta(f))(A) \subseteq \mathbf{N}$ . However, since  $X$  is weakly homogeneous,  $A$  is dense in  $\beta X - X$ . By Theorem 2.3,  $X \cup A$  is pseudocompact as  $X$  is locally compact. Thus  $(\beta(f))(X \cup A) \subseteq \mathbf{N}$ . Since  $X \cup A$  is pseudocompact,  $(\beta(f))(X \cup A)$  is pseudocompact, hence is a bounded subset of  $\mathbf{N}$ . Thus every  $f: X \rightarrow \mathbf{N}$  is a bounded map. This contradicts the fact that there exists an unbounded map from  $X$  to  $\mathbf{N}$ . Thus  $X$  must be  $\mathbf{N}$ -compact.  $\square$

In the above theorem we have used the result, proved by Pierce in [8], that a 0-dimensional space  $X$  is pseudocompact if and only if every continuous integer valued function on  $X$  is bounded.

**2.5 COROLLARY.** *Let  $X$  be 0-dimensional. Suppose the only nontrivial weakly homogeneous subspace of  $\beta X$  containing  $X$  is  $\beta X$  (a trivial subspace is of the form  $\beta X - \{p\}$  where  $p \in \beta X - X$ ). Then  $X$  is pseudocompact.*

**PROOF.** If  $X$  is not pseudocompact then there is a map  $f: X \rightarrow \mathbf{N}$  which is onto. Then  $(\beta(f))^\leftarrow(\mathbf{N})$  is  $\mathbf{N}$ -compact, hence, weakly homogeneous, but is contained properly in  $\beta X$ , and is nontrivial (a trivial subspace—also known as an almost compact subspace—of  $\beta X$  is always pseudocompact as shown in [4, 6J]).  $\square$

In Theorem 2.4 it is obvious that the condition of local compactness on  $X$  is not needed for the implication (ii) implies (i). The question of whether or not local compactness is required in the reverse implication is left open to the reader. If the example given by Nyikos [9] of a realcompact, 0-dimensional space that is not  $\mathbf{N}$ -compact is also weakly homogeneous, then the local compactness condition (or perhaps some weaker condition) on  $X$  is needed in (i) implies (ii).

Finally, it is still an open question whether a realcompact, 0-dimensional, locally compact space is  $\mathbf{N}$ -compact.

#### REFERENCES

1. D. P. Bellamy and L. R. Rubin, *Indecomposable continua in Stone-Ćech compactifications*, Proc. Amer. Math. Soc. **39** (1973), 427–432. MR **47** #4219.
2. R. Engelking and S. Mrówka, *On  $E$ -compact spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **6** (1958), 429–436. MR **20** #3522.
3. N. J. Fine and L. Gillman, *Remote points in  $\beta\mathbf{R}$* , Proc. Amer. Math. Soc. **13** (1962), 29–36. MR **26** #732.
4. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N.J., 1960. MR **22** #6994.

5. H. Herrlich,  *$\mathcal{E}$ -kompakte Räume*, Math. Z. **96** (1967), 228–255. MR **34** #5051.
6. S. Mrówka, *On universal spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **14** (1956), 479–481. MR **19**, 669.
7. ———, *Further results on  $E$ -compact spaces. I*, Acta Math. **120** (1968), 161–185. MR **37** #2165.
8. R. S. Pierce, *Rings of integer-valued continuous functions*, Trans. Amer. Math. Soc. **100** (1961), 371–394. MR **24** #A1289.
9. P. Nyikos, *Prabir Roy's space  $\Delta$  is not  $N$ -compact*, General Topology and Appl. **3** (1973), 197–210. MR **48** #3007.
10. R. C. Walker, *The Stone-Čech compactification*, Ergebnisse Math. Grenzgebiete, Band 83, Springer-Verlag, Berlin and New York, 1974.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG, MANITOBA CANADA

*Current address:* Department of Mathematics, University of Toronto, Toronto, Ontario, Canada  
M5S 1A1