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AN INEQUALITY FOR A CLASS OF INTEGRAL SYSTEMS

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ABSTRACT. A new inequality of the Gronwall type is established for a class of integral systems.

Introduction. The Gronwall Inequality and its generalizations [1], [2], [3] are of fundamental importance in the theory of differential equations. While the bounds provided by these inequalities are adequate in most applications, there are situations where the improved bounds of the following theorem are required.

THEOREM. *Let* K_1, K_2 *and* μ *be nonnegative constants and let* f, g *and* h_i *be continuous nonnegative functions for all* $t \geqslant 0$ *with* h_i *bounded such that*

$$f(t) \leqslant K_1 + \int_0^t h_1(s) f(s)\,ds + \int_0^t e^{\mu s} h_2(s) g(s)\,ds,$$

$$g(t) \leqslant K_2 + \int_0^t e^{-\mu s} h_3(s) f(s)\,ds + \int_0^t h_4(s) g(s)\,ds$$

for all $t \geqslant 0$. *Then there exist constants* c_i *and* M_i *such that*

$$f(t) \leqslant M_1 e^{c_1 t}, \quad g(t) \leqslant M_2 e^{c_2 t}$$

for all $t \geqslant 0$.

PROOF. Suppose $\mu > 0$. Let P be an upper bound for h_i, then

$$f(t) \leqslant K_1 + P\int_0^t f(s)\,ds + P\int_0^t e^{\mu s} g(s)\,ds, \tag{1}$$

$$g(t) \leqslant K_2 + P\int_0^t e^{-\mu s} f(s)\,ds + P\int_0^t g(s)\,ds. \tag{2}$$

Define

$$f_T \equiv \max_{[0,T]} f(t), \quad g_T \equiv \max_{[0,T]} g(t).$$

Since f and g are continuous, f_T and g_T are are attained on $[0, T]$. Therefore

$$(1 - PT) f_T \leqslant K_1 + (P/\mu)(e^{\mu T} - 1) g_T, \tag{3}$$

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(4) $$(1 - PT)g_T \leqslant K_2 + (P/\mu)(1 - e^{-\mu T})f_T.$$

Using (4) in (3) and (3) in (4) we have for $T < 1/P$

(5) $$\left[1 - PT - \frac{P^2}{\mu^2}\frac{(e^{\mu T} - 1)(1 - e^{-\mu T})}{1 - PT}\right]f_T \leqslant K_1 + \frac{P}{\mu}\left(\frac{e^{\mu T} - 1}{1 - PT}\right)K_2,$$

(6) $$\left[1 - PT - \frac{P^2}{\mu^2}\frac{(e^{\mu T} - 1)(1 - e^{-\mu T})}{1 - PT}\right]g_T \leqslant K_2 + \frac{P}{\mu}\left(\frac{1 - e^{-\mu T}}{1 - PT}\right)K_1.$$

We examine the function contained in (5) and (6):

$$H(T) \equiv P + \frac{P^2}{\mu^2}\frac{(e^{\mu T} - 1)(1 - e^{-\mu T})}{T(1 - PT)}.$$

$H(T)$ is continuous for $0 < T < 1/P$, $\lim_{T\to 0^+} H(T) = P$ and $H(T) \geqslant P$. Therefore on the interval $0 < T \leqslant \delta < 1/P$, $H(T)$ has a minimum P and a maximum α. Then for $T < T_1 = \min(\delta, 1/\alpha)$,

(7) $$\frac{1}{1 - PT} \leqslant \frac{1}{1 - H(T)T} \leqslant \frac{1}{1 - \alpha T}.$$

Further there exists a T_2 such that when $0 < T < T_2$,

(8) $$\frac{1}{1 - H(T)T}\frac{P}{\mu}\left(\frac{e^{\mu T} - 1}{1 - PT}\right) \leqslant 1$$

and

(9) $$\frac{1}{1 - H(T)T}\frac{P}{\mu}\left(\frac{1 - e^{-\mu T}}{1 - PT}\right) \leqslant 1.$$

Let $T < \min(T_1, T_2)$ be fixed. Since $0 < 1 - \alpha T < 1$ there exist constants β_i and $\gamma > 0$ such that

(10) $$1 + 1/(1 - \alpha T) \leqslant 1/(1 - \alpha T)^{\beta_1},$$

(11) $$P/\mu + 1/(1 - \alpha T) \leqslant 1/(1 - \alpha T)^{\beta_2},$$

(12) $$1 + P/\mu(1 - \alpha T) \leqslant 1/(1 - \alpha T)^{\beta_3}$$

and

(13) $$1 + 2/(1 - \alpha T)^{\beta} \leqslant 1/(1 - \alpha T)^{\gamma}$$

where $\beta \equiv \max(\beta_1, \beta_2, \beta_3)$. Inequalities (7)–(10) when used in (5) and (6) imply

(14) $$f_T \leqslant \frac{K_1}{1 - H(T)T} + \frac{K_2}{1 - H(T)T}\frac{P}{\mu}\left(\frac{e^{\mu T} - 1}{1 - PT}\right) \leqslant \frac{K}{(1 - \alpha T)^{\beta}},$$

$$(15)\quad g_T \leqslant \frac{K_2}{1 - H(T)T} + \frac{K_1}{1 - H(T)T}\frac{P}{\mu}\left(\frac{1 - e^{-\mu T}}{1 - PT}\right) \leqslant \frac{K}{(1 - \alpha T)^\beta},$$

where $K \equiv \max(K_1, K_2)$.

We consider the interval $[0, 2T]$ and obtain from (1) and (2)

$$(16)\quad (1 - PT)f_{2T} \leqslant K_1 + PTf_T + \frac{P}{\mu}(e^{\mu T} - 1)g_T + \frac{P}{\mu}e^{\mu T}(e^{\mu T} - 1)g_{2T},$$

$$(17)\quad (1 - PT)g_{2T} \leqslant K_2 + PTg_T + \frac{P}{\mu}(1 - e^{-\mu T})f_T + \frac{P}{\mu}e^{-\mu T}(1 - e^{-\mu T})f_{2T}.$$

When (17) is used in (16) and (16) is used in (17) the function $H(T)$ reappears as in (5) and (6). The inequalities (7)–(13) then imply

$$(18)\qquad f_{2T} \leqslant Ke^{\mu T}/(1 - \alpha T)^\delta,$$

$$(19)\qquad g_{2T} \leqslant K/(1 - \alpha T)^\delta,$$

where $\delta = \beta + \gamma$.

Proceeding in the same manner we obtain

$$(20)\qquad f_{nT} \leqslant Ke^{\mu nT}/(1 - \alpha T)^{n\delta},$$

$$(21)\qquad g_{nT} \leqslant K/(1 - \alpha T)^{n\delta}$$

for all integers $n \geqslant 1$. For every $t \geqslant 0$ there exists an integer n such that $(n - 1)T < t \leqslant nT$ and thus

$$(22)\qquad \begin{aligned} f(t) &\leqslant f_{nT} \leqslant Ke^{\mu nT}/(1 - \alpha T)^{n\delta} \\ &\leqslant \frac{K}{(1 - \alpha T)^\delta}\exp\left[\mu T + \left(\mu + \frac{\alpha\delta}{1 - \alpha T}\right)t\right], \end{aligned}$$

$$(23)\qquad g(t) \leqslant g_{nT} \leqslant \frac{K}{(1 - \alpha T)^{n\delta}} \leqslant \frac{K}{(1 - \alpha T)^\delta}\exp\left(\frac{\alpha\delta t}{1 - \alpha T}\right)$$

for all $t \geqslant 0$. This result uses the identity $(1 - \alpha T)^{-1} = 1 + \alpha T(1 - \alpha T)^{-1}$ and the inequality $(1 + 1/n)^n < e$. A similar argument may be carried out starting with (1) and (2) to show that these bounds also hold for $\mu = 0$ when α, δ and T are properly chosen. Equations (22) and (23) provide the constants c_i and M_i.

EXAMPLE. The system in equations (1) and (2)

$$f(t) \leqslant K_1 + P\int_0^t f(s)\,ds + P\int_0^t e^{\mu s}g(s)\,ds,$$

$$g(t) \leqslant K_2 + P\int_0^t e^{-\mu s}f(s)\,ds + P\int_0^t g(s)\,ds$$

arises in the study of kinetic model equations [4]. One would like to establish that the functions f and g are at most of exponential order so that the Laplace transform may be applied. Both the Gronwall Inequality [3] and the comparison theorems of Nohel [5], [6] provide bounds on f and g of exponential functions raised to exponential functions. The inequalities (22) and (23), on the other hand, establish that these functions are of exponential order. The sharp inequalities of this new theorem are produced by the novel iteration carried out in the proof.

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