

AN INEQUALITY FOR A CLASS OF INTEGRAL SYSTEMS

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ABSTRACT. A new inequality of the Gronwall type is established for a class of integral systems.

Introduction. The Gronwall Inequality and its generalizations [1], [2], [3] are of fundamental importance in the theory of differential equations. While the bounds provided by these inequalities are adequate in most applications, there are situations where the improved bounds of the following theorem are required.

THEOREM. Let K_1, K_2 and μ be nonnegative constants and let f, g and h_i be continuous nonnegative functions for all $t \geq 0$ with h_i bounded such that

$$f(t) \leq K_1 + \int_0^t h_1(s)f(s)ds + \int_0^t e^{\mu s} h_2(s)g(s)ds,$$

$$g(t) \leq K_2 + \int_0^t e^{-\mu s} h_3(s)f(s)ds + \int_0^t h_4(s)g(s)ds$$

for all $t \geq 0$. Then there exist constants c_i and M_i such that

$$f(t) \leq M_1 e^{c_1 t}, \quad g(t) \leq M_2 e^{c_2 t}$$

for all $t \geq 0$.

PROOF. Suppose $\mu > 0$. Let P be an upper bound for h_i , then

$$(1) \quad f(t) \leq K_1 + P \int_0^t f(s)ds + P \int_0^t e^{\mu s} g(s)ds,$$

$$(2) \quad g(t) \leq K_2 + P \int_0^t e^{-\mu s} f(s)ds + P \int_0^t g(s)ds.$$

Define

$$f_T \equiv \max_{[0, T]} f(t), \quad g_T \equiv \max_{[0, T]} g(t).$$

Since f and g are continuous, f_T and g_T are attained on $[0, T]$. Therefore

$$(3) \quad (1 - PT)f_T \leq K_1 + (P/\mu)(e^{\mu T} - 1)g_T,$$

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$$(4) \quad (1 - PT)g_T \leq K_2 + (P/\mu)(1 - e^{-\mu T})f_T.$$

Using (4) in (3) and (3) in (4) we have for $T < 1/P$

$$(5) \quad \left[1 - PT - \frac{P^2 (e^{\mu T} - 1)(1 - e^{-\mu T})}{\mu^2 (1 - PT)} \right] f_T \leq K_1 + \frac{P}{\mu} \left(\frac{e^{\mu T} - 1}{1 - PT} \right) K_2,$$

$$(6) \quad \left[1 - PT - \frac{P^2 (e^{\mu T} - 1)(1 - e^{-\mu T})}{\mu^2 (1 - PT)} \right] g_T \leq K_2 + \frac{P}{\mu} \left(\frac{1 - e^{-\mu T}}{1 - PT} \right) K_1.$$

We examine the function contained in (5) and (6):

$$H(T) \equiv P + \frac{P^2 (e^{\mu T} - 1)(1 - e^{-\mu T})}{\mu^2 T(1 - PT)}.$$

$H(T)$ is continuous for $0 < T < 1/P$, $\lim_{T \rightarrow 0^+} H(T) = P$ and $H(T) \geq P$. Therefore on the interval $0 < T \leq \delta < 1/P$, $H(T)$ has a minimum P and a maximum α . Then for $T < T_1 = \min(\delta, 1/\alpha)$,

$$(7) \quad \frac{1}{1 - PT} \leq \frac{1}{1 - H(T)T} \leq \frac{1}{1 - \alpha T}.$$

Further there exists a T_2 such that when $0 < T < T_2$,

$$(8) \quad \frac{1}{1 - H(T)T} \frac{P}{\mu} \left(\frac{e^{\mu T} - 1}{1 - PT} \right) \leq 1$$

and

$$(9) \quad \frac{1}{1 - H(T)T} \frac{P}{\mu} \left(\frac{1 - e^{-\mu T}}{1 - PT} \right) \leq 1.$$

Let $T < \min(T_1, T_2)$ be fixed. Since $0 < 1 - \alpha T < 1$ there exist constants β_i and $\gamma > 0$ such that

$$(10) \quad 1 + 1/(1 - \alpha T) \leq 1/(1 - \alpha T)^{\beta_1},$$

$$(11) \quad P/\mu + 1/(1 - \alpha T) \leq 1/(1 - \alpha T)^{\beta_2},$$

$$(12) \quad 1 + P/\mu(1 - \alpha T) \leq 1/(1 - \alpha T)^{\beta_3}$$

and

$$(13) \quad 1 + 2/(1 - \alpha T)^\beta \leq 1/(1 - \alpha T)^\gamma$$

where $\beta \equiv \max(\beta_1, \beta_2, \beta_3)$. Inequalities (7)–(10) when used in (5) and (6) imply

$$(14) \quad f_T \leq \frac{K_1}{1 - H(T)T} + \frac{K_2}{1 - H(T)T} \frac{P}{\mu} \left(\frac{e^{\mu T} - 1}{1 - PT} \right) \leq \frac{K}{(1 - \alpha T)^\beta},$$

$$(15) \quad g_T \leq \frac{K_2}{1 - H(T)T} + \frac{K_1}{1 - H(T)T} \frac{P}{\mu} \left(\frac{1 - e^{-\mu T}}{1 - PT} \right) \leq \frac{K}{(1 - \alpha T)^\beta},$$

where $K \equiv \max(K_1, K_2)$.

We consider the interval $[0, 2T]$ and obtain from (1) and (2)

$$(16) \quad (1 - PT)f_{2T} \leq K_1 + PTf_T + \frac{P}{\mu}(e^{\mu T} - 1)g_T + \frac{P}{\mu}e^{\mu T}(e^{\mu T} - 1)g_{2T},$$

$$(17) \quad (1 - PT)g_{2T} \leq K_2 + PTg_T + \frac{P}{\mu}(1 - e^{-\mu T})f_T + \frac{P}{\mu}e^{-\mu T}(1 - e^{-\mu T})f_{2T}.$$

When (17) is used in (16) and (16) is used in (17) the function $H(T)$ reappears as in (5) and (6). The inequalities (7)–(13) then imply

$$(18) \quad f_{2T} \leq Ke^{\mu T}/(1 - \alpha T)^\delta,$$

$$(19) \quad g_{2T} \leq K/(1 - \alpha T)^\delta,$$

where $\delta = \beta + \gamma$.

Proceeding in the same manner we obtain

$$(20) \quad f_{nT} \leq Ke^{\mu nT}/(1 - \alpha T)^{n\delta},$$

$$(21) \quad g_{nT} \leq K/(1 - \alpha T)^{n\delta}$$

for all integers $n \geq 1$. For every $t \geq 0$ there exists an integer n such that $(n - 1)T < t \leq nT$ and thus

$$(22) \quad \begin{aligned} f(t) &\leq f_{nT} \leq Ke^{\mu nT}/(1 - \alpha T)^{n\delta} \\ &\leq \frac{K}{(1 - \alpha T)^\delta} \exp\left[\mu T + \left(\mu + \frac{\alpha\delta}{1 - \alpha T}\right)t\right], \end{aligned}$$

$$(23) \quad g(t) \leq g_{nT} \leq \frac{K}{(1 - \alpha T)^{n\delta}} \leq \frac{K}{(1 - \alpha T)^\delta} \exp\left(\frac{\alpha\delta t}{1 - \alpha T}\right)$$

for all $t \geq 0$. This result uses the identity $(1 - \alpha T)^{-1} = 1 + \alpha T(1 - \alpha T)^{-1}$ and the inequality $(1 + 1/n)^n < e$. A similar argument may be carried out starting with (1) and (2) to show that these bounds also hold for $\mu = 0$ when α , δ and T are properly chosen. Equations (22) and (23) provide the constants c_i and M_i .

EXAMPLE. The system in equations (1) and (2)

$$f(t) \leq K_1 + P \int_0^t f(s) ds + P \int_0^t e^{\mu s} g(s) ds,$$

$$g(t) \leq K_2 + P \int_0^t e^{-\mu s} f(s) ds + P \int_0^t g(s) ds$$

arises in the study of kinetic model equations [4]. One would like to establish that the functions f and g are at most of exponential order so that the Laplace transform may be applied. Both the Gronwall Inequality [3] and the comparison theorems of Nohel [5], [6] provide bounds on f and g of exponential functions raised to exponential functions. The inequalities (22) and (23), on the other hand, establish that these functions are of exponential order. The sharp inequalities of this new theorem are produced by the novel iteration carried out in the proof.

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