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ASYMPTOTIC NONBASES WHICH ARE NOT SUBSETS OF MAXIMAL ASYMPTOTIC NONBASES

JULIEN HENNEFELD

ABSTRACT. Let A be a set of positive integers. If all but a finite number of the positive integers can be written as a sum of h elements of A, then A is called an asymptotic basis of order h. Otherwise, A is called an asymptotic nonbasis of order h. For each $h \geqslant 2$, we construct an asymptotic nonbasis of order h which is not a subset of a maximal asymptotic nonbasis of order h.

For A, a set of positive integers, let hA denote the set of all sums of h not necessarily distinct elements of A. If hA contains all but finitely many of the positive integers, then A is an asymptotic basis of order h. If A is an asymptotic nonbasis of order h, but $A \cup \{a\}$ is an asymptotic basis of order h for every positive integer $a \notin A$, then A is called a maximal asymptotic nonbasis of order h.

The question of whether every asymptotic nonbasis of order h is a subset of some maximal asymptotic nonbasis of order h was originally posed by Nathanson [3] and repeated by Erdös and Nathanson [1], [2].

THEOREM. *For $h \geqslant 2$, let $A = \{1\} \cup \{h\} \cup$ {all multiples of h, except $q_1, q_2, q_3, \ldots$}, where $\{q_i\}$ is an increasing sequence of multiples of h, with $\lim(q_{i+1} - q_i) = \infty$. Then A is an asymptotic nonbasis of order h which is not a subset of any maximal asymptotic nonbasis of order h.*

PROOF. To simplify the notation, we will only give the proof for $h > 3$. Clearly, since 1 is the only element of A not congruent to 0 $(\operatorname{mod} h)$, hA will miss all integers of the form $q_i + h - 1$. Also, note that all but finitely many of the integers that hA misses are of the form $q_i + h - 1$. This is because for all i sufficiently large,

$$q_i + h - 2 = (q_i - h) + h + 1 + \cdots + 1$$

is the sum of h elements of A. Also,

$$q_i + h - 3 = (q_i - 2h) + h + h + 1 + \cdots + 1$$

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is the sum of h elements of A. Similarly, all integers in $[q_i, q_i + h - 2]$ have representations as sums of h elements in A. Every integer in $[q_i + h, q_{i+1}]$ is in hA for i sufficiently large, because each $q_i + rh$ in (q_i, q_{i+1}) can be expressed as a sum of j elements from A for each j in $[1, h]$.

We claim that if x is any integer greater than 1, which is not congruent to $0 \pmod{h}$, then $A \cup \{x\}$ is an asymptotic basis of order h. To prove this claim we will consider separately the two cases: $x \equiv 1 \pmod{h}$ and $x \not\equiv 1 \pmod{h}$.

If $x \equiv 1 \pmod{h}$, then for all i sufficiently large,

$$q_i + 1 = x + [q_i - (x - 1)]$$

is the sum of two elements of $A \cup \{x\}$, and thus by $h - 2$ additions of 1, $q_i + h - 1$ will be the sum of h elements of $A \cup \{x\}$.

If $x \not\equiv 1 \pmod{h}$, take m large enough so that $hx < q_i - q_{i-1}$, for all $i \geqslant m$. Pick p: $ph < x < (p + 1)h$. Then $q_i + 1 < (q_i - ph) + x < q_i + h$. Since $(q_i - ph) + x$ is the sum of two elements from $A \cup \{x\}$, $q_i + h - 1$ can be expressed as a sum of fewer than h elements of $A \cup \{x\}$. To express it as a sum of exactly h elements, use an expression of the form

$$[q_i - (p + r)h] + x + h + \cdots + h + 1 + \cdots + 1.$$

Thus the claim is proved.

Therefore, the only hope for extending A to a maximal asymptotic nonbasis is by adjoining B, a subset of $\{q_1, q_2, \ldots\}$. However, if there are infinitely many q_i missing from B, then $A \cup B$ will be a nonbasis which is not maximal. If there are only finitely many q_i missing from B, then $A \cup B$ will be a basis. In neither case will $A \cup B$ be a maximal nonbasis.

REMARK. The author originally constructed the example $A = \{1\} \cup \{$all multiples of h, except $h^2, h^3, h^4, \ldots\}$ and wishes to thank M. B. Nathanson for pointing out the somewhat more general example given in this paper.

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DEPARTMENT OF MATHEMATICS, BROOKLYN COLLEGE, CUNY, BROOKLYN, NEW YORK 11210