ASYMPTOTIC NONBASES WHICH ARE NOT SUBSETS OF MAXIMAL ASYMPTOTIC NONBASES

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Abstract. Let $A$ be a set of positive integers. If all but a finite number of the positive integers can be written as a sum of $h$ elements of $A$, then $A$ is called an asymptotic basis of order $h$. Otherwise, $A$ is called an asymptotic nonbasis of order $h$. For each $h \geq 2$, we construct an asymptotic nonbasis of order $h$ which is not a subset of a maximal asymptotic nonbasis of order $h$.

For $A$, a set of positive integers, let $hA$ denote the set of all sums of $h$ not necessarily distinct elements of $A$. If $hA$ contains all but finitely many of the positive integers, then $A$ is an asymptotic basis of order $h$. If $A$ is an asymptotic nonbasis of order $h$, but $A \cup \{a\}$ is an asymptotic basis of order $h$ for every positive integer $a \in A$, then $A$ is called a maximal asymptotic nonbasis of order $h$.

The question of whether every asymptotic nonbasis of order $h$ is a subset of some maximal asymptotic nonbasis of order $h$ was originally posed by Nathanson [3] and repeated by Erdös and Nathanson [1], [2].

Theorem. For $h \geq 2$, let $A = \{1\} \cup \{h\} \cup \{\text{all multiples of } h, \text{ except } q_1, q_2, q_3, \ldots\}$, where $\{q_i\}$ is an increasing sequence of multiples of $h$, with $\lim(q_{i+1} - q_i) = \infty$. Then $A$ is an asymptotic nonbasis of order $h$ which is not a subset of any maximal asymptotic nonbasis of order $h$.

Proof. To simplify the notation, we will only give the proof for $h \geq 3$. Clearly, since 1 is the only element of $A$ not congruent to 0 (mod $h$), $hA$ will miss all integers of the form $q_i + h - 1$. Also, note that all but finitely many of the integers that $hA$ misses are of the form $q_i + h - 1$. This is because for all $i$ sufficiently large,

$$q_i + h - 2 = (q_i - h) + h + 1 + \cdots + 1$$

is the sum of $h$ elements of $A$. Also,

$$q_i + h - 3 = (q_i - 2h) + h + h + 1 + \cdots + 1$$
is the sum of \( h \) elements of \( A \). Similarly, all integers in \( [q_i, q_i + h - 2] \) have representations as sums of \( h \) elements in \( A \). Every integer in \( [q_i + h, q_{i+1}] \) is in \( hA \) for \( i \) sufficiently large, because each \( q_i + rh \) in \( (q_i, q_{i+1}) \) can be expressed as a sum of \( j \) elements from \( A \) for each \( j \) in \([1, h]\).

We claim that if \( x \) is any integer greater than 1, which is not congruent to 0 (mod \( h \)), then \( A \cup \{x\} \) is an asymptotic basis of order \( h \). To prove this claim we will consider separately the two cases: \( x \equiv 1 \) (mod \( h \)) and \( x \not\equiv 1 \) (mod \( h \)).

If \( x \equiv 1 \) (mod \( h \)), then for all \( i \) sufficiently large,

\[
q_i + 1 = x + [q_i - (x - 1)]
\]

is the sum of two elements of \( A \cup \{x\} \), and thus by \( h - 2 \) additions of 1, \( q_i + h - 1 \) will be the sum of \( h \) elements of \( A \cup \{x\} \).

If \( x \not\equiv 1 \) (mod \( h \)), take \( m \) large enough so that \( hx < q_i - q_{i-1} \), for all \( i \geq m \). Pick \( p \): \( ph < x < (p + 1)h \). Then \( q_i + 1 < (q_i - ph) + x < q_i + h \). Since \( (q_i - ph) + x \) is the sum of two elements from \( A \cup \{x\} \), \( q_i + h - 1 \) can be expressed as a sum of fewer than \( h \) elements of \( A \cup \{x\} \). To express it as a sum of exactly \( h \) elements, use an expression of the form

\[
[q_i - (p + r)h] + x + h + \cdots + h + 1 + \cdots + 1.
\]

Thus the claim is proved.

Therefore, the only hope for extending \( A \) to a maximal asymptotic nonbasis is by adjoining \( B \), a subset of \( \{q_1, q_2, \ldots\} \). However, if there are infinitely many \( q_i \) missing from \( B \), then \( A \cup B \) will be a nonbasis which is not maximal. If there are only finitely many \( q_i \) missing from \( B \), then \( A \cup B \) will be a basis. In neither case will \( A \cup B \) be a maximal nonbasis.

**Remark.** The author originally constructed the example \( A = \{1\} \cup \{\text{all multiples of } h, \text{except } h^2, h^3, h^4, \ldots\} \) and wishes to thank M. B. Nathanson for pointing out the somewhat more general example given in this paper.

**References**


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