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ON THE INVERSES OF M-MATRICES

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ABSTRACT. If A is the adjugate of an M-matrix, the sign of a certain class of minors of A is characterized by a simple rule. These minors, to be called 'near principal minors', properly contain the so-called 'almost principal minors'.

I. **Notation.** For an arbitrary $n \times n$ matrix $A = (a_{ij})$, $A^T = (a_{ij}^t)$ will denote the transpose of A and A^{adj} the classical adjugate of A, that is the matrix whose entries are the $(n-1)$-rowed signed minors of A.[1] The minor of A with row-indices $r_1, r_2, \ldots, r_p$ and column indices $c_1, c_2, \ldots, c_p$, where $1 \leqslant r_1 < r_2 < \cdots < r_p \leqslant n$ and $1 \leqslant c_1 < c_2 < \cdots < c_p \leqslant n$ will be denoted by

$$A\begin{pmatrix} r_1 & r_2 \cdots r_p \\ c_1 & c_2 \cdots c_p \end{pmatrix}.$$

II. **Introduction.** An M-matrix is a square matrix whose off-diagonal entries are nonpositive and whose principal minors are nonnegative.[2] A nonnegative matrix is one whose entries are nonnegative. The Stieltjes-Ostrowski Theorem [5], [7] says that the adjugate of an M-matrix is nonnegative. An interesting question is: Which nonnegative matrices have adjugates or inverses that are M-matrices? Markham [3] has given a partial answer to this. He defines an almost principal minor of a matrix A, in agreement with Gantmaher [2, p. 102], as a minor with the property that exactly one of the row-column differences $r_1 - c_1, r_2 - c_2, \ldots, r_p - c_p$ is nonzero. It is shown in his paper that if A is nonnegative of order n, and A^{-1} is an M-matrix, then the almost principal minors of A are nonnegative. A natural generalization of the concept 'almost principal minor' allows a wider result which is the object of this paper and contains Markham's result as a corollary.

III. **The result.** The term 'near principal minor' of a matrix A will mean a minor where, with exactly one exception, each row index is equal to some column index and conversely. The exceptional row index will be denoted by r, and the exceptional column index by c. The pair (r, c) will be called the unmatched pair. These terms will also be applied to square submatrices of A.

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[1]Notation follows [1, p. 52].

[2]This definition is essentially equivalent to that given by Markham [3]. For the elementary proof of this, see [6, p. 206].

EXAMPLE. The minor $A\binom{1\ 3\ 8\ 9\ 12}{1\ 3\ 6\ 8\ 12}$ is near principal, but not almost principal. The unmatched pair is (9, 6).

THEOREM. *Let A be the adjugate of an M-matrix B, and A_m an m-rowed near-principal submatrix of A with unmatched pair (r, c). The sign of the near principal minor* $\det A_m$ *is $(-)^s$ where s = (number of row indices of $A_m < r$) + (number of column indices of $A_m < c$), except in the trivial case* $\det A_m = 0$.

COMMENTS. The near-principal minor of the example has three row indices < 9 and two column indices < 6, so that $s = 5$. Now if A is the adjugate of an M-matrix, the theorem predicts that the sign of this minor will be $(-)^5 = (-)$.

For a near principal minor that is actually almost principal, s will be even. Thus the almost principal minors of the adjugate of an M-matrix are guaranteed by the theorem to be nonnegative. This is essentially Markham's result.

PROOF OF THE THEOREM. Let B^T_{n-m} denote the $(n - m)$-rowed submatrix of B^T whose row and column indices are complementary to those of A_m. With A_m, B^T_{n-m} is near principal, but has unmatched pair (c, r).

Jacobi's Theorem [1, p. 97] says that any minor of order m in $A = B^{\text{adj}}$ is equal to the complementary signed minor in B^T multiplied by $(\det B)^{m-1}$. Application of this to $\det A_m$ yields

$$(-)^{c+r}(\det B^T_{n-m})(\det B)^{m-1} = \det A_m.$$

On the other hand, B^T_{n-m} is a submatrix of a principal submatrix B^T_{n-m+1} having $n - m + 1$ rows. Since B is an M-matrix, B^T and B^T_{n-m+1} will be too. The Stieltjes-Ostrowski Theorem applied to the matrix B^T_{n-m+1} predicts that $\det B^T_{n-m}$, multiplied by the algebraic sign it receives as the cofactor of b^t_{rc} in $\det B^T_{n-m+1}$ is nonnegative. This sign is not merely $(-)^{r+c}$, since this would not correspond to a consecutive numbering of the rows and columns of B^T_{n-m+1}. Rather, it is $(-)^{r'+c'}$ where r' is the number of rows of B^T_{n-m+1} with index $\leqslant r$, and c' is the number of columns of B^T_{n-m+1} with index $\leqslant c$. Multiplication of both sides of the above equation by $(-)^{r'+c'}$ yields

$$(-)^{r+c}(-)^{r'+c'}(\det B^T_{n-m})(\det B)^{m-1} = (-)^{r'+c'}\det A_m.$$

Since $(-)^{r'+c'} \det B^T_{n-m} \geqslant 0$ by the Stieltjes-Ostrowski Theorem, and $\det B \geqslant 0$ by assumption, it follows that

$$\text{sign of } \det A_m = (-)^{r-r'+c-c'}.$$

However, because the indices of A_m are complementary to those of B^T_{n-m}, it follows that r' + (number of row indices of $A_m < r$) $= r$. Thus $r - r'$ = (number of row indices of $A_m < r$). Similarly $c - c'$ = (number of column indices of $A_m < c$). This says that $r - r' + c - c' = s$, from which the theorem follows.

IV. **Concluding remarks**. The theorem can be viewed as a generalization of the Stieltjes-Ostrowski Theorem, since each entry of A is a near principal minor of order 1 whose sign, according to the rule given, must be $(+)$ if A is the adjugate of an M-matrix.

If the determinant of an M-matrix is positive (> 0), then each of its principal minors is positive as well [6, p. 187]. If, in addition, the off-diagonal entries are negative (< 0), an alternate version of the Stieltjes-Ostrowski Theorem says that each entry of the adjugate of the matrix is positive [4, p. 206]. If these assumptions are made about the M-matrix B of the theorem, and this form of the Stieltjes-Ostrowski Theorem is applied in the above proof, it can be shown that the sign rule given in the Theorem holds strictly, i.e. that each near principal minor of A has the 'right' sign and that none vanishes.

A matrix A with positive determinant is the adjugate of an M-matrix if and only if A^{adj} is an M-matrix. This is because

$$A = \left(\frac{A^{\mathrm{adj}}}{(\det A)^{(n-2)/(n-1)}} \right)^{\mathrm{adj}}$$

if $\det A > 0$. The theorem of §III was actually designed to help characterize those matrices whose adjugates are M-matrices. Now if the principal minors of A are nonnegative, satisfaction of the sign rule given in the theorem is sufficient to guarantee that A^{adj} is an M-matrix, and also, if $\det A > 0$, that A is the adjugate of an M-matrix. This is a kind of converse of the Stieltjes-Ostrowski Theorem, but not a desirable one, because verifying the rule (indeed even just for near principal minors of order $n - 1$) involves computing the adjugate of A. Is it possible to decide that A^{adj} is an M-matrix by checking the signs of relatively few near principal minors of A? This problem is unsolved.

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References

1. A. C. Aitken, *Determinants and matrices*, Oliver and Boyd, Edinburgh, 1956.

2. F. R. Gantmaher, *The theory of matrices*, GITTL, Moscow, 1953; English transl., Vol. 2, Chelsea, New York, 1959. MR **21** #6372c.

3. T. L. Markham, *Nonnegative matrices whose inverses are m-matrices*, Proc. Amer. Math. Soc. **36** (1972), 326–330. MR **46** #9073.

4. B. Murphy, *Expansion of* $(n - 1)$*-rowed subdeterminants*, Math. Z. **147** (1976), 205–206.

5. A. Ostrowski, *Über die Determinanten mit überwiegender Hauptdiagonale*, Comment. Math. Helv. **10** (1937–1938), 69–96.

6. ———, *Determinanten mit überwiegender Hauptdiagonale und die absolute Konvergenz von linearen Iterationsprozessen*, Comment. Math. Helv. **30** (1956), 175–210. MR **17**, 898.

7. T. J. Stieltjes, *Sur les racines de l'equation* $X_n = 0$, Acta Math. **9** (1886), 385–400.

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