

ON THE INVERSES OF M -MATRICES

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ABSTRACT. If A is the adjugate of an M -matrix, the sign of a certain class of minors of A is characterized by a simple rule. These minors, to be called 'near principal minors', properly contain the so-called 'almost principal minors'.

I. Notation. For an arbitrary $n \times n$ matrix $A = (a_{ij})$, $A^T = (a_{ij}^t)$ will denote the transpose of A and A^{adj} the classical adjugate of A , that is the matrix whose entries are the $(n - 1)$ -rowed signed minors of A .¹ The minor of A with row-indices r_1, r_2, \dots, r_p and column indices c_1, c_2, \dots, c_p , where $1 \leq r_1 < r_2 < \dots < r_p \leq n$ and $1 \leq c_1 < c_2 < \dots < c_p \leq n$ will be denoted by

$$A \begin{pmatrix} r_1 & r_2 & \dots & r_p \\ c_1 & c_2 & \dots & c_p \end{pmatrix}.$$

II. Introduction. An M -matrix is a square matrix whose off-diagonal entries are nonpositive and whose principal minors are nonnegative.² A nonnegative matrix is one whose entries are nonnegative. The Stieltjes-Ostrowski Theorem [5], [7] says that the adjugate of an M -matrix is nonnegative. An interesting question is: Which nonnegative matrices have adjugates or inverses that are M -matrices? Markham [3] has given a partial answer to this. He defines an almost principal minor of a matrix A , in agreement with Gantmaher [2, p. 102], as a minor with the property that exactly one of the row-column differences $r_1 - c_1, r_2 - c_2, \dots, r_p - c_p$ is nonzero. It is shown in his paper that if A is nonnegative of order n , and A^{-1} is an M -matrix, then the almost principal minors of A are nonnegative. A natural generalization of the concept 'almost principal minor' allows a wider result which is the object of this paper and contains Markham's result as a corollary.

III. The result. The term 'near principal minor' of a matrix A will mean a minor where, with exactly one exception, each row index is equal to some column index and conversely. The exceptional row index will be denoted by r , and the exceptional column index by c . The pair (r, c) will be called the unmatched pair. These terms will also be applied to square submatrices of A .

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¹Notation follows [1, p. 52].

²This definition is essentially equivalent to that given by Markham [3]. For the elementary proof of this, see [6, p. 206].

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EXAMPLE. The minor $A \begin{pmatrix} 1 & 3 & 8 & 9 & 12 \\ 1 & 3 & 6 & 8 & 12 \end{pmatrix}$ is near principal, but not almost principal. The unmatched pair is (9, 6).

THEOREM. Let A be the adjugate of an M -matrix B , and A_m an m -rowed near-principal submatrix of A with unmatched pair (r, c) . The sign of the near principal minor $\det A_m$ is $(-)^s$ where $s = (\text{number of row indices of } A_m < r) + (\text{number of column indices of } A_m < c)$, except in the trivial case $\det A_m = 0$.

COMMENTS. The near-principal minor of the example has three row indices < 9 and two column indices < 6 , so that $s = 5$. Now if A is the adjugate of an M -matrix, the theorem predicts that the sign of this minor will be $(-)^5 = (-)$.

For a near principal minor that is actually almost principal, s will be even. Thus the almost principal minors of the adjugate of an M -matrix are guaranteed by the theorem to be nonnegative. This is essentially Markham's result.

PROOF OF THE THEOREM. Let B_{n-m}^T denote the $(n - m)$ -rowed submatrix of B^T whose row and column indices are complementary to those of A_m . With A_m , B_{n-m}^T is near principal, but has unmatched pair (c, r) .

Jacobi's Theorem [1, p. 97] says that any minor of order m in $A = B^{\text{adj}}$ is equal to the complementary signed minor in B^T multiplied by $(\det B)^{m-1}$. Application of this to $\det A_m$ yields

$$(-)^{c+r} (\det B_{n-m}^T) (\det B)^{m-1} = \det A_m.$$

On the other hand, B_{n-m}^T is a submatrix of a principal submatrix B_{n-m+1}^T having $n - m + 1$ rows. Since B is an M -matrix, B^T and B_{n-m+1}^T will be too. The Stieltjes-Ostrowski Theorem applied to the matrix B_{n-m+1}^T predicts that $\det B_{n-m}^T$, multiplied by the algebraic sign it receives as the cofactor of b_{rc}' in $\det B_{n-m+1}^T$ is nonnegative. This sign is not merely $(-)^{r+c}$, since this would not correspond to a consecutive numbering of the rows and columns of B_{n-m+1}^T . Rather, it is $(-)^{r'+c'}$ where r' is the number of rows of B_{n-m+1}^T with index $\leq r$, and c' is the number of columns of B_{n-m+1}^T with index $\leq c$. Multiplication of both sides of the above equation by $(-)^{r'+c'}$ yields

$$(-)^{r+c} (-)^{r'+c'} (\det B_{n-m}^T) (\det B)^{m-1} = (-)^{r'+c'} \det A_m.$$

Since $(-)^{r'+c'} \det B_{n-m}^T \geq 0$ by the Stieltjes-Ostrowski Theorem, and $\det B \geq 0$ by assumption, it follows that

$$\text{sign of } \det A_m = (-)^{r-r'+c-c'}.$$

However, because the indices of A_m are complementary to those of B_{n-m}^T , it follows that $r' + (\text{number of row indices of } A_m < r) = r$. Thus $r - r' = (\text{number of row indices of } A_m < r)$. Similarly $c - c' = (\text{number of column indices of } A_m < c)$. This says that $r - r' + c - c' = s$, from which the theorem follows.

IV. **Concluding remarks.** The theorem can be viewed as a generalization of the Stieltjes-Ostrowski Theorem, since each entry of A is a near principal

minor of order 1 whose sign, according to the rule given, must be (+) if A is the adjugate of an M -matrix.

If the determinant of an M -matrix is positive (> 0), then each of its principal minors is positive as well [6, p. 187]. If, in addition, the off-diagonal entries are negative (< 0), an alternate version of the Stieltjes-Ostrowski Theorem says that each entry of the adjugate of the matrix is positive [4, p. 206]. If these assumptions are made about the M -matrix B of the theorem, and this form of the Stieltjes-Ostrowski Theorem is applied in the above proof, it can be shown that the sign rule given in the Theorem holds strictly, i.e. that each near principal minor of A has the 'right' sign and that none vanishes.

A matrix A with positive determinant is the adjugate of an M -matrix if and only if A^{adj} is an M -matrix. This is because

$$A = \left(\frac{A^{\text{adj}}}{(\det A)^{(n-2)/(n-1)}} \right)^{\text{adj}}$$

if $\det A > 0$. The theorem of §III was actually designed to help characterize those matrices whose adjugates are M -matrices. Now if the principal minors of A are nonnegative, satisfaction of the sign rule given in the theorem is sufficient to guarantee that A^{adj} is an M -matrix, and also, if $\det A > 0$, that A is the adjugate of an M -matrix. This is a kind of converse of the Stieltjes-Ostrowski Theorem, but not a desirable one, because verifying the rule (indeed even just for near principal minors of order $n - 1$) involves computing the adjugate of A . Is it possible to decide that A^{adj} is an M -matrix by checking the signs of relatively few near principal minors of A ? This problem is unsolved.

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