

GENERATORS AND RELATIONS FOR THE SPECIAL LINEAR GROUP OVER A DIVISION RING¹

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ABSTRACT. Let \mathfrak{R} be a division ring, $n \geq 3$ an integer, and let $SL(n, \mathfrak{R})$ be the special linear group over \mathfrak{R} . In this paper a presentation in terms of generators and relations is given for $SL(n, \mathfrak{R})$.

Let \mathfrak{R} be a division ring, \mathfrak{R}^* its group of units. If G is a group and $a, b \in G$ we write $(a, b) = aba^{-1}b^{-1}$ for the commutator of a and b and (G, G) for the commutator subgroup of G . If n is an integer, let $GL(n, \mathfrak{R})$ denote the general linear group consisting of all $n \times n$ invertible matrices, and let $SL(n, \mathfrak{R})$ denote the special linear group consisting of all $n \times n$ matrices of determinant 1 (in the sense of Dieudonné). If i and j are distinct integers between 1 and n and $u \in \mathfrak{R}$, let $e_{ij}(u) \in SL(n, \mathfrak{R})$ denote the matrix with entry u in the (i, j) th place, ones on the diagonal, and zeros elsewhere. It is well known that the $e_{ij}(u)$ generate $SL(n, \mathfrak{R})$.

If $u \in \mathfrak{R}^*$ we define elements $m_{ij}(u)$ and $d_{ij}(u)$ of $SL(n, \mathfrak{R})$ to be

$$(1) \quad m_{ij}(u) = e_{ij}(u)e_{ji}(-u^{-1})e_{ij}(u)$$

and

$$(2) \quad d_{ij}(u) = m_{ij}(u)m_{ij}(-1).$$

Note that $d_{ij}(u) = \text{diag}(1, \dots, 1, u, 1, \dots, 1, u^{-1}, 1, \dots, 1)$, the diagonal matrix with u in the (i, i) th place and u^{-1} in the (j, j) th place.

If $u, v \in \mathfrak{R}^*$ we define the element $a_1(u, v)$ of $SL(n, \mathfrak{R})$ to be

$$(3) \quad a_1(u, v) = d_{12}(u)d_{12}(v)d_{12}(vu)^{-1}.$$

Note that $a_1(u, v) = \text{diag}((u, v), 1, \dots, 1)$, the diagonal matrix with (u, v) in the $(1, 1)$ st place.

We will now give a description of generators and relations for $SL(n, \mathfrak{R})$.

THEOREM. *If $n \geq 3$, $SL(n, \mathfrak{R})$ is generated by the symbols $e_{ij}(u)$, where i and j are distinct integers between 1 and n and $u \in \mathfrak{R}$, subject to the relations*

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$$(A) \quad e_{ij}(u)e_{ij}(v) = e_{ij}(u + v),$$

$$(B) \quad (e_{ij}(u), e_{kl}(v)) = \begin{cases} 1 & \text{if } j \neq k, i \neq l, \\ e_{il}(uv) & \text{if } j = k, i \neq l, \\ e_{kj}(-vu) & \text{if } j \neq k, i = l. \end{cases}$$

Let $m_{ij}(u), d_{ij}(u), a_1(u, v)$ be as in (1), (2), and (3).

$$(C) \quad a_1(u, v) = d_{ij}(u)d_{ij}(v)d_{ij}(vu)^{-1}$$

if $j \neq 1$ and

$$a_1(u, v) = d_{ij}(u)d_{ij}(v)d_{ij}(vu)^{-1} d_{1i}((u, v))$$

if 1, i, j are distinct.

$$(D) \quad \prod_{j=1}^s a_1(u_j, v_j)^{\epsilon_j} = 1 \quad \text{if } \prod_{j=1}^s (u_j, v_j)^{\epsilon_j} = 1, \epsilon_j = \pm 1.$$

PROOF. Certainly (A), (B), (C), (D) hold in $SL(n, \mathfrak{R})$.

Now let $St(n, \mathfrak{R})$ be the Steinberg group generated by the symbols $x_{ij}(u)$, where i and j are distinct integers between 1 and n and $u \in \mathfrak{R}$, subject to the relations obtained by replacing the $e_{ij}(u)$ by $x_{ij}(u)$ in (A), (B). For $u, v \in \mathfrak{R}^*$, we define elements $w_{ij}(u), h_{ij}(u)$, and $b(u, v)$ of $St(n, \mathfrak{R})$ as

$$w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u),$$

$$h_{ij}(u) = w_{ij}(u)w_{ij}(-1),$$

$$b(u, v) = h_{12}(u)h_{12}(v)h_{12}(vu)^{-1}.$$

If i and j are distinct between 1 and n and $u \in \mathfrak{R}$ we consider symbols $\bar{x}_{ij}(u)$. Define $\bar{w}_{ij}(u), \bar{h}_{ij}(u), \bar{b}(u, v)$ in terms of the symbols $\bar{x}_{ij}(u)$ as we defined $w_{ij}(u), h_{ij}(u), b(u, v)$ in terms of $x_{ij}(u)$. Let G be the group generated by the symbols $\bar{x}_{ij}(u)$ subject to relations $(\bar{A}), (\bar{B}), (\bar{C}), (\bar{D})$, obtained from (A), (B), (C), (D) by replacing $e_{ij}(u), m_{ij}(u), d_{ij}(u), a_1(u, v)$ by $\bar{x}_{ij}(u), \bar{w}_{ij}(u), \bar{h}_{ij}(u), \bar{b}(u, v)$, respectively.

Therefore we have epimorphisms $St(n, \mathfrak{R}) \xrightarrow{\varphi} G \xrightarrow{\alpha} SL(n, \mathfrak{R})$ defined by $\varphi(x_{ij}(u)) = \bar{x}_{ij}(u)$ and $\alpha(\bar{x}_{ij}(u)) = e_{ij}(u)$. We wish to prove α is an isomorphism.

Let W be the subgroup of $St(n, \mathfrak{R})$ generated by the $w_{ij}(u)$, H the subgroup generated by the $h_{ij}(u)$, and B the subgroup generated by the $b(u, v)$. Let \bar{W}, \bar{H} , and \bar{B} be the corresponding subgroups of G .

LEMMA 1. *The kernel of $\alpha\varphi$ is contained in H . Hence the kernel of α is contained in \bar{H} .*

PROOF. By [1, p. 78, Theorem 9.12], $\ker(\alpha\phi) \subset W$, and hence is equal to the kernel of $\alpha\phi|_W: W \rightarrow \text{SL}(n, \mathbb{R})$. As in the proof of [1, p. 77, Theorem 9.11] we have $\ker(\alpha\phi) \subset H$. The second statement follows immediately from the first.

LEMMA 2. Every element of \bar{H} can be written in the form $\bar{b} \prod_{i=1}^{n-1} \bar{h}_{i,i+1}(u_i)$ where $\bar{b} \in \bar{B}$.

PROOF. Let H_i be the subgroup of H generated by the $h_{i,i+1}(u)$. Then by [1, p. 72, Corollary 9.4] H_i is normalized by all $h_{kl}(u)$, so $H_1 H_2 \cdots H_{n-1}$ is a subgroup. We will now show $H = H_1 H_2 \cdots H_{n-1}$, that is, that $H_1 \cdots H_{n-1}$ contains all $h_{kl}(u)$, $k < l$. If $l = k + 1$, it is clear, and we proceed by induction on $l - k$. Choose p so that $k < p < l$. Then by [1, p. 76, Lemma 9.10] $h_{kl}(u) = h_{pl}(u)h_{kp}(u)$, and each of these is in $H_1 H_2 \cdots H_{n-1}$ by induction. Therefore $H = H_1 H_2 \cdots H_{n-1}$. Now if \bar{H}_i is the subgroup of \bar{H} generated by the $\bar{h}_{i,i+1}(u)$, we have that $\bar{H} = \bar{H}_1 \bar{H}_2 \cdots \bar{H}_{n-1}$.

By a similar argument, $\bar{H}_1 \bar{H}_2 \cdots \bar{H}_{i-1}$ is the subgroup generated by all $\bar{h}_{kl}(u)$, $k < l \leq i$.

If $\bar{h} \in \bar{H}$, $\bar{h} = \bar{h}_1 \bar{h}_2 \cdots \bar{h}_{n-1}$ where $\bar{h}_i \in \bar{H}_i$, since $H = H_1 H_2 \cdots H_{n-1}$.

Claim. If $i \neq 1$, $\bar{h}_i = \bar{h} \bar{h}_{i,i+1}(u)$ for some $u \in \mathbb{R}^*$ and some $\bar{h}' \in \bar{H}_1 \bar{H}_2 \cdots \bar{H}_{i-1}$.

Suppose

$$\bar{h}_i = \bar{h}_{i,i+1}(u_1)^{\epsilon_1} \bar{h}_{i,i+1}(u_2)^{\epsilon_2} \cdots \bar{h}_{i,i+1}(u_r)^{\epsilon_r}, \quad r \geq 1, \epsilon_i = \pm 1.$$

Since $\bar{h}_{i,i+1}(u)^{-1} = \bar{b}(u^{-1}, u)^{-1} \bar{h}_{i,i+1}(u^{-1})$, by (\bar{C}) , we may assume $\epsilon_1 = 1, r > 1$, and

$$h_i = \bar{h}_{i,i+1}(u_1) \bar{h}_{i,i+1}(u_2)^{\epsilon_2} \cdots \bar{h}_{i,i+1}(u_r)^{\epsilon_r}.$$

If $\epsilon_2 = 1$ we have

$$\bar{h}_i = \bar{b}(u_1, u_2) \bar{h}_i((u_1, u_2))^{-1} \bar{h}_{i,i+1}(u_2 u_1) \bar{h}_{i,i+1}(u_3)^{\epsilon_3} \cdots \bar{h}_{i,i+1}(u_r)^{\epsilon_r}$$

by (\bar{C}) , and $\bar{b}(u_1, u_2) \bar{h}_i((u_1, u_2))^{-1} \in \bar{H}_1 \bar{H}_2 \cdots \bar{H}_{i-1}$. If $\epsilon_2 = -1$ we have

$$\begin{aligned} \bar{h}_{i,i+1}(u_1) \bar{h}_{i,i+1}(u_2)^{-1} &= (\bar{h}_{i,i+1}(u_2) \bar{h}_{i,i+1}(u_1)^{-1})^{-1} \\ &= [\bar{h}_{i,i+1}(u_2^{-1} u_1)^{-1} \bar{b}(u_2^{-1} u_1, u_2) \bar{h}_i((u_2^{-1}, u_1))^{-1}]^{-1} \\ &= \bar{h}_i((u_2^{-1}, u_1)) \bar{b}(u_2^{-1} u_1, u_2)^{-1} \bar{h}_{i,i+1}(u_2^{-1} u_1) \end{aligned}$$

by (\bar{C}) . Hence in this case

$$\bar{h}_i = \bar{h}_i((u_2^{-1}, u_1)) \bar{b}(u_2^{-1} u_1, u_2)^{-1} \bar{h}_{i,i+1}(u_2^{-1} u_1) \bar{h}_{i,i+1}(u_3)^{\epsilon_3} \cdots \bar{h}_{i,i+1}(u_r)^{\epsilon_r},$$

and $\bar{h}_i((u_2^{-1}, u_1)) \bar{b}(u_2^{-1} u_1, u_2)^{-1} \in \bar{H}_1 \bar{H}_2 \cdots \bar{H}_{i-1}$.

By induction on r and since $\bar{H}_1 \bar{H}_2 \cdots \bar{H}_{i-1}$ is a subgroup we have that $\bar{h}_i = \bar{h} \bar{h}_{i,i+1}(u)$ as desired.

By the claim we get

$$\bar{h} = \bar{h}'_1 \bar{h}_{23}(u_3) \cdots \bar{h}_{n-1,n}(u_n)$$

where $\bar{h}'_1 \in H_1$. Using induction and the definition of $\bar{b}(u, v)$, we have $\bar{h}'_1 = \bar{b} \bar{h}_{12}(u_2)$ so that

$$\bar{h} = \bar{b} \bar{h}_{12}(u_2) \cdots \bar{h}_{n-1,n}(u_n).$$

We will now complete the proof of the Theorem by showing that α is an isomorphism. Let $x \in \ker \alpha$. By Lemma 1 $x \in \bar{H}$, so by Lemma 2

$$x = \bar{b} \prod_{i=1}^{n-1} \bar{h}_{i,i+1}(u_i).$$

Applying α we obtain

$$1 = a \prod_{i=1}^{n-1} d_{i,i+1}(u_i)$$

where $a \in A$, the subgroup generated by all $a_1(u, v)$, $u, v \in \mathfrak{R}^*$. Therefore if $i \geq 2$, $u_{i-1}^{-1} u_i = 1$, and $u_{n-1}^{-1} = 1$, which implies $u_i = 1$ for $i = 1, 2, \dots, n - 1$. Hence

$$x = \bar{b} = \prod_{j=1}^s \bar{b}(u_j, v_j)^{\epsilon_j}$$

for some $u_j, v_j \in \mathfrak{R}^*$, $\epsilon_j = \pm 1$. Applying α to this expression we obtain

$$1 = \prod_{j=1}^s a_1(u_j, v_j)^{\epsilon_j},$$

which implies $\prod_{j=1}^s (u_j, v_j)^{\epsilon_j} = 1$, and $x = \bar{b} = 1$ by (D). Therefore α is an isomorphism.

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