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GENERATORS AND RELATIONS FOR THE SPECIAL LINEAR GROUP OVER A DIVISION RING[1]

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ABSTRACT. Let $\mathfrak{R}$ be a division ring, $n \geqslant 3$ an integer, and let $\mathrm{SL}(n, \mathfrak{R})$ be the special linear group over $\mathfrak{R}$. In this paper a presentation in terms of generators and relations is given for $\mathrm{SL}(n, \mathfrak{R})$.

Let $\mathfrak{R}$ be a division ring, $\mathfrak{R}^*$ its group of units. If G is a group and $a, b \in G$ we write $(a, b) = aba^{-1}b^{-1}$ for the commutator of a and b and (G, G) for the commutator subgroup of G. If n is an integer, let $\mathrm{GL}(n, \mathfrak{R})$ denote the general linear group consisting of all $n \times n$ invertible matrices, and let $\mathrm{SL}(n, \mathfrak{R})$ denote the special linear group consisting of all $n \times n$ matrices of determinant 1 (in the sense of Dieudonné). If i and j are distinct integers between 1 and n and $u \in \mathfrak{R}$, let $e_{ij}(u) \in \mathrm{SL}(n, \mathfrak{R})$ denote the matrix with entry u in the (i,j)th place, ones on the diagonal, and zeros elsewhere. It is well known that the $e_{ij}(u)$ generate $\mathrm{SL}(n, \mathfrak{R})$.

If $u \in \mathfrak{R}^*$ we define elements $m_{ij}(u)$ and $d_{ij}(u)$ of $\mathrm{SL}(n, \mathfrak{R})$ to be

$$m_{ij}(u) = e_{ij}(u)e_{ji}(-u^{-1})e_{ij}(u) \tag{1}$$

and

$$d_{ij}(u) = m_{ij}(u)m_{ij}(-1). \tag{2}$$

Note that $d_{ij}(u) = \mathrm{diag}(1, \ldots, 1, u, 1, \ldots, 1, u^{-1}, 1, \ldots, 1)$, the diagonal matrix with u in the (i, i)th place and u^{-1} in the (j,j)th place.

If $u, v \in \mathfrak{R}^*$ we define the element $a_1(u, v)$ of $\mathrm{SL}(n, \mathfrak{R})$ to be

$$a_1(u, v) = d_{12}(u)d_{12}(v)d_{12}(vu)^{-1}. \tag{3}$$

Note that $a_1(u, v) = \mathrm{diag}((u, v), 1, \ldots, 1)$, the diagonal matrix with (u, v) in the $(1, 1)$st place.

We will now give a description of generators and relations for $\mathrm{SL}(n, \mathfrak{R})$.

THEOREM. *If $n \geqslant 3$, $\mathrm{SL}(n, \mathfrak{R})$ is generated by the symbols $e_{ij}(u)$, where i and j are distinct integers between* 1 *and n and $u \in \mathfrak{R}$, subject to the relations*

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(A) $$e_{ij}(u)e_{ij}(v) = e_{ij}(u+v),$$

(B) $$(e_{ij}(u), e_{kl}(v)) = \begin{cases} 1 & \text{if } j \neq k,\ i \neq l, \\ e_{il}(uv) & \text{if } j = k,\ i \neq l, \\ e_{kj}(-vu) & \text{if } j \neq k,\ i = l. \end{cases}$$

Let $m_{ij}(u)$, $d_{ij}(u)$, $a_1(u,v)$ *be as in* (1), (2), *and* (3).

(C) $$a_1(u,v) = d_{1j}(u)d_{1j}(v)d_{1j}(vu)^{-1}$$

if $j \neq 1$ *and*

$$a_1(u,v) = d_{ij}(u)d_{ij}(v)d_{ij}(vu)^{-1}d_{1i}((u,v))$$

if 1, i, j *are distinct.*

(D) $$\prod_{j=1}^{s} a_1(u_j, v_j)^{\varepsilon_j} = 1 \quad \text{if } \prod_{j=1}^{s} (u_j, v_j)^{\varepsilon_j} = 1,\ \varepsilon_j = \pm 1.$$

PROOF. Certainly (A), (B), (C), (D) hold in $SL(n, \mathfrak{R})$.

Now let $St(n, \mathfrak{R})$ be the Steinberg group generated by the symbols $x_{ij}(u)$, where i and j are distinct integers between 1 and n and $u \in \mathfrak{R}$, subject to the relations obtained by replacing the $e_{ij}(u)$ by $x_{ij}(u)$ in (A), (B). For $u, v \in \mathfrak{R}^*$, we define elements $w_{ij}(u)$, $h_{ij}(u)$, and $b(u,v)$ of $St(n, \mathfrak{R})$ as

$$w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u),$$

$$h_{ij}(u) = w_{ij}(u)w_{ij}(-1),$$

$$b(u,v) = h_{12}(u)h_{12}(v)h_{12}(vu)^{-1}.$$

If i and j are distinct between 1 and n and $u \in \mathfrak{R}$ we consider symbols $\bar{x}_{ij}(u)$. Define $\bar{w}_{ij}(u)$, $\bar{h}_{ij}(u)$, $\bar{b}(u,v)$ in terms of the symbols $\bar{x}_{ij}(u)$ as we defined $w_{ij}(u)$, $h_{ij}(u)$, $b(u,v)$ in terms of $x_{ij}(u)$. Let G be the group generated by the symbols $\bar{x}_{ij}(u)$ subject to relations $(\bar{A})$, $(\bar{B})$, $(\bar{C})$, $(\bar{D})$, obtained from (A), (B), (C), (D) by replacing $e_{ij}(u)$, $m_{ij}(u)$, $d_{ij}(u)$, $a_1(u,v)$ by $\bar{x}_{ij}(u)$, $\bar{w}_{ij}(u)$, $\bar{h}_{ij}(u)$, $\bar{b}(u, v)$, respectively.

Therefore we have epimorphisms $St(n, \mathfrak{R}) \xrightarrow{\varphi} G \xrightarrow{\alpha} SL(n, \mathfrak{R})$ defined by $\varphi(x_{ij}(u)) = \bar{x}_{ij}(u)$ and $\alpha(\bar{x}_{ij}(u)) = e_{ij}(u)$. We wish to prove α is an isomorphism.

Let W be the subgroup of $St(n, \mathfrak{R})$ generated by the $w_{ij}(u)$, H the subgroup generated by the $h_{ij}(u)$, and B the subgroup generated by the $b(u,v)$. Let $\bar{W}$, $\bar{H}$, and $\bar{B}$ be the corresponding subgroups of G.

LEMMA 1. *The kernel of* $\alpha\varphi$ *is contained in* H. *Hence the kernel of* α *is contained in* $\bar{H}$.

PROOF. By [1, p. 78, Theorem 9.12], $\ker(\alpha\phi) \subset W$, and hence is equal to the kernel of $\alpha\varphi|_W \colon W \to \mathrm{SL}(n, \mathfrak{R})$. As in the proof of [1, p. 77, Theorem 9.11] we have $\ker(\alpha\varphi) \subset H$. The second statement follows immediately from the first.

LEMMA 2. *Every element of* $\overline{H}$ *can be written in the form* $\overline{b}\prod_{i=1}^{n-1}\overline{h}_{i,i+1}(u_i)$ *where* $\overline{b} \in \overline{B}$.

PROOF. Let H_i be the subgroup of H generated by the $h_{i,i+1}(u)$. Then by [1, p. 72, Corollary 9.4] H_i is normalized by all $h_{kl}(u)$, so $H_1 H_2 \cdots H_{n-1}$ is a subgroup. We will now show $H = H_1 H_2 \cdots H_{n-1}$, that is, that $H_1 \cdots H_{n-1}$ contains all $h_{kl}(u)$, $k < l$. If $l = k+1$, it is clear, and we proceed by induction on $l - k$. Choose p so that $k < p < l$. Then by [1, p. 76, Lemma 9.10] $h_{kl}(u) = h_{pl}(u)h_{kp}(u)$, and each of these is in $H_1 H_2 \cdots H_{n-1}$ by induction. Therefore $H = H_1 H_2 \cdots H_{n-1}$. Now if $\overline{H}_i$ is the subgroup of $\overline{H}$ generated by the $\overline{h}_{i,i+1}(u)$, we have that $\overline{H} = \overline{H}_1\overline{H}_2 \cdots \overline{H}_{n-1}$.

By a similar argument, $\overline{H}_1\overline{H}_2 \cdots \overline{H}_{i-1}$ is the subgroup generated by all $\overline{h}_{kl}(u)$, $k < l \leqslant i$.

If $\overline{h} \in \overline{H}$, $\overline{h} = \overline{h}_1\overline{h}_2 \cdots \overline{h}_{n-1}$ where $\overline{h}_i \in \overline{H}_i$, since $\overline{H} = \overline{H}_1\overline{H}_2 \cdots \overline{H}_{n-1}$.

Claim. If $i \neq 1$, $\overline{h}_i = \overline{h}'\overline{h}_{i,i+1}(u)$ for some $u \in \mathfrak{R}^*$ and some $\overline{h}' \in \overline{H}_1\overline{H}_2 \cdots \overline{H}_{i-1}$.

Suppose

$$\overline{h}_i = \overline{h}_{i,i+1}(u_1)^{\varepsilon_1}\overline{h}_{i,i+1}(u_2)^{\varepsilon_2} \cdots \overline{h}_{i,i+1}(u_r)^{\varepsilon_r}, \qquad r \geqslant 1,\ \varepsilon_i = \pm 1.$$

Since $\overline{h}_{i,i+1}(u)^{-1} = \overline{b}(u^{-1}, u)^{-1}\overline{h}_{i,i+1}(u^{-1})$, by $(\overline{\mathrm{C}})$, we may assume $\varepsilon_1 = 1$, $r > 1$, and

$$h_i = \overline{h}_{i,i+1}(u_1)\overline{h}_{i,i+1}(u_2)^{\varepsilon_2} \cdots \overline{h}_{i,i+1}(u_r)^{\varepsilon_r}.$$

If $\varepsilon_2 = 1$ we have

$$\overline{h}_i = \overline{b}(u_1, u_2)\overline{h}_{1i}((u_1, u_2))^{-1}\overline{h}_{i,i+1}(u_2 u_1)\overline{h}_{i,i+1}(u_3)^{\varepsilon_3} \cdots \overline{h}_{i,i+1}(u_r)^{\varepsilon_r}$$

by $(\overline{\mathrm{C}})$, and $\overline{b}(u_1, u_2)\overline{h}_{1i}((u_1, u_2))^{-1} \in \overline{H}_1\overline{H}_2 \cdots \overline{H}_{i-1}$. If $\varepsilon_2 = -1$ we have

$$\begin{aligned}\overline{h}_{i,i+1}(u_1)\overline{h}_{i,i+1}(u_2)^{-1} &= (\overline{h}_{i,i+1}(u_2)\overline{h}_{i,i+1}(u_1)^{-1})^{-1}\\ &= [\overline{h}_{i,i+1}(u_2^{-1}u_1)^{-1}\overline{b}(u_2^{-1}u_1, u_2)\overline{h}_{1i}((u_2^{-1}, u_1))^{-1}]^{-1}\\ &= \overline{h}_{1i}((u_2^{-1}, u_1))\overline{b}(u_2^{-1}u_1, u_2)^{-1}\overline{h}_{i,i+1}(u_2^{-1}u_1)\end{aligned}$$

by $(\overline{\mathrm{C}})$. Hence in this case

$$\overline{h}_i = \overline{h}_{1i}((u_2^{-1}, u_1))\overline{b}(u_2^{-1}u_1, u_2)^{-1}\overline{h}_{i,i+1}(u_2^{-1}u_1)\overline{h}_{i,i+1}(u_3)^{\varepsilon_3} \cdots \overline{h}_{i,i+1}(u_r)^{\varepsilon_r},$$

and $\overline{h}_{1i}((u_2^{-1}, u_1))\overline{b}(u_2^{-1}u_1, u_2)^{-1} \in \overline{H}_1\overline{H}_2 \cdots \overline{H}_{i-1}$.

By induction on r and since $\overline{H}_1\overline{H}_2\cdots\overline{H}_{i-1}$ is a subgroup we have that $\overline{h}_i = \overline{h}'\overline{h}_{i,i+1}(u)$ as desired.

By the claim we get

$$\overline{h} = \overline{h}_1'\overline{h}_{23}(u_3)\cdots\overline{h}_{n-1,n}(u_n)$$

where $\overline{h}_1' \in H_1$. Using induction and the definition of $\overline{b}(u,v)$, we have $\overline{h}_1' = \overline{b}\overline{h}_{12}(u_2)$ so that

$$\overline{h} = \overline{b}\overline{h}_{12}(u_2)\cdots\overline{h}_{n-1,n}(u_n).$$

We will now complete the proof of the Theorem by showing that α is an isomorphism. Let $x \in \ker\alpha$. By Lemma 1 $x \in \overline{H}$, so by Lemma 2

$$x = \overline{b}\prod_{i=1}^{n-1}\overline{h}_{i,i+1}(u_i).$$

Applying α we obtain

$$1 = a\prod_{i=1}^{n-1}d_{i,i+1}(u_i)$$

where $a \in A$, the subgroup generated by all $a_1(u,v)$, u, $v \in \mathfrak{R}^*$. Therefore if $i \geqslant 2$, $u_{i-1}^{-1}u_i = 1$, and $u_{n-1}^{-1} = 1$, which implies $u_i = 1$ for $i = 1, 2, \ldots, n-1$. Hence

$$x = \overline{b} = \prod_{j=1}^{s}\overline{b}(u_j, v_j)^{\varepsilon_j}$$

for some u_j, $v_j \in \mathfrak{R}^*$, $\varepsilon_j = \pm 1$. Applying α to this expression we obtain

$$1 = \prod_{j=1}^{s}a_1(u_j, v_j)^{\varepsilon_j},$$

which implies $\prod_{j=1}^{s}(u_j, v_j)^{\varepsilon_j} = 1$, and $x = \overline{b} = 1$ by $(\overline{\mathrm{D}})$. Therefore α is an isomorphism.

Bibliography

1. J. Milnor, *Introduction to algebraic K-theory*, Ann. of Math. Studies, no. 72, Princeton Univ. Press, Princeton, N.J.; Univ. of Tokyo Press, Tokyo, 1971. MR **50** #2304.

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