

BOUNDEDNESS OF VALUE REGIONS AND CONVERGENCE OF CONTINUED FRACTIONS

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ABSTRACT. If the elements of a continued fraction are restricted to lie within some region E of the complex plane, it is quite often possible to determine, with very little difficulty, where the approximants of the continued fraction lie. Generally, it is more difficult to determine whether every continued fraction with elements from this set E is convergent. In this paper, we give some results which, in certain cases, reduce the question of convergence to the question of whether the set of approximants is bounded.

1. We will be concerned with two forms of the continued fraction

$$(1.1) \quad \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots$$

where each a_p and b_p is a complex number. We say that (1.1) is of type (A) provided that each b_p has value 1 and that it is of type (B) provided that each a_p has value 1. For certain computations, we will use the fundamental recurrence relations:

$$(1.2) \quad \begin{aligned} A_{-1} &= B_0 = 1, & B_{-1} &= A_0 = 0; \\ A_p &= b_p A_{p-1} + a_p A_{p-2}; \\ B_p &= b_p B_{p-1} + a_p B_{p-2}, & p &= 1, 2, 3, \dots \end{aligned}$$

It follows that for $p = 1, 2, 3, \dots$,

$$(1.3) \quad \frac{A_p}{B_p} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_p}{b_p}$$

and

$$(1.4) \quad A_{p-1}B_p - A_pB_{p-1} = (-1)^p a_1 a_2 \cdots a_p.$$

As is customary, we say that (1.1) converges provided that there are not infinitely many values of p such that $B_p = 0$ and the sequence $\{A_p/B_p\}$ converges to a finite limit.

If E is a set of complex numbers, we use the symbol $V_A(E)$ to denote the set of all complex numbers of the form

$$\frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_n}{1}$$

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where each one of a_1, a_2, \dots, a_n lies in E ; the symbol $V_B(E)$ denotes the set of all complex numbers of the form

$$\frac{1}{b_1} + \frac{1}{b_2} + \dots + \frac{1}{b_n}$$

where each one of b_1, b_2, \dots, b_n lies in E . We say that the set E of complex numbers is a type (A) convergence set provided that if a_p lies in E and $b_p = 1$, $p = 1, 2, 3, \dots$, (1.1) converges; we say that it is a type (B) convergence set provided that if b_p lies in E and $a_p = 1$, $p = 1, 2, 3, \dots$, (1.1) converges. If M is a set of complex numbers, the symbols $M + M$ and $M \cdot M$ denote, respectively, the set of all sums $x + y$ and all products xy with x in M and y in M .

2. In this section, we will prove the following two theorems.

THEOREM A. *Suppose that E is a set of complex numbers such that -1 is not a limit point of $V_A(E) + V_A(E)$. Then E is a type (A) convergence set if and only if $V_A(E)$ is bounded.*

THEOREM B. *Suppose that E is a set of complex numbers such that -1 is not a limit point of $V_B(E) \cdot V_B(E)$. Then E is a type (B) convergence set if and only if $V_B(E)$ is bounded.*

As would be expected, the proofs of these theorems are similar. We will give a proof of Theorem A and indicate, at certain points, the modifications which can be made to obtain a proof of Theorem B.

Suppose that E is a set of complex numbers such that -1 is not a limit point of $V_A(E) + V_A(E)$ and $V_A(E)$ is bounded. Under this hypothesis, no B_p has value 0. To see that this is true, suppose that for some p , $B_p = 0$ and let k denote the least such value of p . From (1.2), $B_1 = 1$ and $B_2 = 1 + a_2$. If B_2 were 0, then -1 would be in E and hence, $-1/(1 + (-1/1)) = \infty$ would be in $V_A(E)$. Therefore, $k > 2$. But $B_k = B_{k-1} + a_k B_{k-2} = 0$ so that $a_k = -B_{k-1}/B_{k-2}$. By (1.2),

$$a_k = -\left(1 + \frac{a_{k-1}}{B_{k-2}/B_{k-3}}\right).$$

Continuing in this manner, we find that

$$a_k = -\left(1 + \frac{a_{k-1}}{1} + \frac{a_{k-2}}{1} + \dots + \frac{a_1}{1}\right).$$

By a simple calculation, we see that if a is any complex number distinct from 0,

$$\frac{a}{1} + \frac{a_1}{1} + \frac{a_2}{1} + \dots + \frac{a_{k-1}}{1} + \frac{a_k}{1} = \infty.$$

Therefore, no B_p has value 0.

With regard to Theorem B, we see that 0 is not in E and since $A_2/B_2 = b_2/(b_1 b_2 + 1)$, $B_2 \neq 0$. From the equation $B_k = b_k B_{k-1} + B_{k-2} = 0$, it follows that

$$b_k = \frac{1}{-b_{k-1}} + \frac{1}{-b_{k-2}} + \dots + \frac{1}{-b_1}$$

and hence,

$$\frac{1}{b_1} + \frac{1}{b_2} + \dots + \frac{1}{b_{k-1}} + \frac{1}{b_k} = \infty.$$

The number -1 does not belong to $V_A(E) + V_A(E)$, for if it did, then there would be a sequence $a_1, a_2, \dots, a_n, a'_1, a'_2, \dots, a'_m$ from E such that

$$\frac{a_1}{1} + \frac{a_2}{1} + \dots + \frac{a_n}{1} = -1 - \left(\frac{a'_1}{1} + \frac{a'_2}{1} + \dots + \frac{a'_m}{1} \right).$$

Then for any nonzero a , we would have

$$\frac{a}{1} + \frac{a'_m}{1} + \dots + \frac{a'_1}{1} + \frac{a_1}{1} + \dots + \frac{a_n}{1} = \infty.$$

For Theorem B, if

$$\left(\frac{1}{b_1} + \frac{1}{b_2} + \dots + \frac{1}{b_n} \right) \cdot \left(\frac{1}{b'_1} + \frac{1}{b'_2} + \dots + \frac{1}{b'_m} \right) = -1,$$

then

$$\frac{1}{b'_m} + \frac{1}{b'_{m-1}} + \dots + \frac{1}{b'_1} + \frac{1}{b_1} + \dots + \frac{1}{b_n} = \infty.$$

Also, under the hypothesis above, there exist positive numbers a and b such that if each of x and y is in $V_A(E)$, $a \leq |1 + x + y| \leq b$. Therefore, there is a positive number M such that if each x, y, x' , and y' is in $V_A(E)$, then

$$(2.1) \quad M|1 + x + y| \leq |1 + x' + y'|.$$

Suppose that each one of a_1, a_2, a_3, \dots lies in E . For $p = 1, 2, 3, \dots$, let $t_p(z) = a_p/(1 + z)$ and $s_p(z) = t_1 t_2 \dots t_p(z)$ so that $s_p(0) = A_p/B_p$. Let n_1, n_2, n_3, \dots and k_1, k_2, k_3, \dots be increasing sequences of positive integers such that $n_p + 2 < k_p < n_{p+1}, p = 1, 2, 3, \dots$. Then, for $p = 1, 2, 3, \dots$, we have

$$\begin{aligned} M \cdot \left| \left(1 + \frac{a_{n_p+1}}{1} + \dots + \frac{a_1}{1} \right) + \left(\frac{a_{n_p+2}}{1} + \dots + \frac{a_{n_{p+1}}}{1} \right) \right| \\ \leq \left| \left(1 + \frac{a_{n_p+1}}{1} + \dots + \frac{a_1}{1} \right) + \left(\frac{a_{n_p+2}}{1} + \dots + \frac{a_{k_p}}{1} \right) \right| \end{aligned}$$

and, referring to (1.2),

$$\frac{M}{|B_{n_p+1}/B_{n_p} + t_{n_p+2} \dots t_{k_p}(0)|} \leq \frac{1}{|B_{n_p+1}/B_{n_p} + t_{n_p+2} \dots t_{n_{p+1}}(0)|}.$$

If we multiply by $|a_1 a_2 \dots a_{n_p}|/|B_{n_p}|^2$ and make use of (1.2), we obtain

$$\frac{M|a_1 a_2 \dots a_{n_p}| |x|}{|B_{n_p}| |B_{n_p} + x B_{n_p-1}|} \leq \frac{|a_1 a_2 \dots a_{n_p}| |y|}{|B_{n_p}| |B_{n_p} + y B_{n_p-1}|}$$

where

$$x = \frac{a_{n_p+1}}{1} + \dots + \frac{a_{k_p}}{1} \quad \text{and} \quad y = \frac{a_{n_p+1}}{1} + \dots + \frac{a_{n_{p+1}}}{1}.$$

However, this is the same as

$$(2.2) \quad M|s_{n_p}(0) - s_{k_p}(0)| \leq |s_{n_p}(0) - s_{n_{p+1}}(0)|.$$

Since $V_A(E)$ is bounded, some subsequence of $\{s_p(0)\}_{p=1}^\infty$ is convergent and, with (2.2), we see that no two subsequences converge to different limits. Therefore, E is a type (A) convergence set.

The proof for Theorem B is quite similar. Rather than (2.1) we have

$$(2.3) \quad M|1 + xy| \leq |1 + x'y'|.$$

If we now suppose that for $p = 1, 2, 3, \dots$, b_p lies in E , $t_p(z) = 1/(b_p + z)$, and $s_p(z) = t_1 t_2 \dots t_p(z)$, we have

$$\frac{M}{|1 + t_{n_p+1} \dots t_{k_p}(0)B_{n_p}/B_{n_{p+1}}|} \leq \frac{1}{|1 + t_{n_p+1} \dots t_{n_{p+1}}(0)B_{n_p}/B_{n_{p-1}}|}.$$

Multiplying by $|B_{n_p}| |B_{n_{p+1}}|$ and making use of (1.2), we have, as before, (2.2).

Now suppose that E is a set of complex numbers such that -1 is not a limit point of $V_A(E) + V_A(E)$ and that E is a type (A) convergence set. If $V_A(E)$ were not bounded, there would exist a sequence $a_1^1, a_1^2, a_2^2, a_3^2, a_2^3, a_3^3, \dots$, with each term in E such that

$$(2.4) \quad \frac{a_1^1}{1}, \frac{a_1^2}{1 + \frac{1}{1}}, \frac{a_2^2}{1 + \frac{1}{1}}, \frac{a_1^3}{1 + \frac{1}{1 + \frac{1}{1}}}, \frac{a_2^3}{1 + \frac{1}{1 + \frac{1}{1}}}, \frac{a_3^3}{1 + \frac{1}{1 + \frac{1}{1}}}, \dots$$

converges to ∞ . As is well known, in order that E be a type (A) convergence set it is necessary that it be bounded so that the sequence $a_1^1, a_1^2, a_1^3, \dots$ does not increase without bound and, hence, the sequence of reciprocals of (2.4) must converge to 0. From this it follows that the sequence

$$\frac{a_2^2}{1}, \frac{a_2^3}{1 + \frac{1}{1}}, \frac{a_3^3}{1 + \frac{1}{1}}, \frac{a_2^4}{1 + \frac{1}{1 + \frac{1}{1}}}, \frac{a_3^4}{1 + \frac{1}{1 + \frac{1}{1}}}, \frac{a_4^4}{1 + \frac{1}{1 + \frac{1}{1}}}, \dots$$

converges to -1 . For any nonzero complex number a , the sequence obtained by applying the transformation $a/(1 + z)$ to each term of (2.4) converges to 0. If such a sequence is added term by term to the sequence above, we see that -1 is a limit point of $V_A(E) + V_A(E)$.

In the case of Theorem B, E must be bounded away from 0 and from the assumption that

$$\frac{1}{b_1^1}, \frac{1}{b_1^2 + \frac{1}{b_2^2}}, \frac{1}{b_1^3 + \frac{1}{b_2^3 + \frac{1}{b_3^3}}}, \dots$$

converges to ∞ , it follows that

$$\frac{1}{b_1^2} \cdot \frac{1}{b_2^2}, \frac{1}{b_1^3} \cdot \left(\frac{1}{b_2^3 + \frac{1}{b_3^3}} \right), \frac{1}{b_1^4} \cdot \left(\frac{1}{b_2^4 + \frac{1}{b_3^4 + \frac{1}{b_4^4}}} \right), \dots$$

converges to -1 .

3. For real numbers, we have the following stronger results.

THEOREM A'. *Suppose that E is a set of real numbers. Then E is a type (A) convergence set if and only if $V_A(E)$ is bounded.*

THEOREM B'. *Suppose that E is a set of real numbers. Then E is a type (B) convergence set if and only if $V_B(E)$ is bounded.*

Suppose that a is a real number less than $-1/4$. Let u denote $1/2 + i(-1 - 4a)^{1/2}/2$ and let v denote $u^2/|u|^2$. If, for $p = 1, 2, 3, \dots$, $a_p = a$ and $b_p = 1$, then

$$\frac{A_n}{B_n} = - \frac{\bar{u}}{v + 1/\sum_{p=0}^{n-1} v^p}$$

(see [4, p. 23]). The denominator of this expression can be written as $v + (1 - v)/(1 - v^n)$. If v is a root of unity, then for some n , $v^n = 1/v$, in which case this denominator is 0. If v is not a root of unity, the sequence $\{v^n\}$ is dense in the unit circle and hence some subsequence of these denominators converges to 0. From these considerations, we see that if $V_A(E)$ is bounded, E does not contain a real number less than $-1/4$. Since E must be a subset of $V_A(E)$, when $V_A(E)$ is bounded, so is E . However, it is well known that if E is a bounded set of real numbers and no element of E is less than $-1/4$, then E is a type (A) convergence set (see, for example, [4, Theorem 14.2, p. 58]). It is also easily shown that if each element of the bounded set E is a real number not less than $-1/4$, then $V_A(E)$ is bounded. If E contains an element less than $-1/4$ or is unbounded, it is not a type (A) convergence set. Thus, we see that Theorem A' is true.

Suppose that p and q are numbers such that $-4/p < q < 0 < p$. Then $1/pq < -1/4$. Since

$$\frac{1}{b_1} + \frac{1}{b_2} + \dots + \frac{1}{b_n} = \frac{a_1}{1} + \frac{a_2}{1} + \dots + \frac{a_n}{1}$$

where $a_1 = 1/b_1$ and $a_p = 1/b_{p-1}b_p$, $p = 2, 3, 4, \dots$, it is not possible for both p and q to belong to E and $V_A(E)$ to be bounded. If $V_B(E)$ is bounded, then E is bounded away from 0. If there exists a positive number p such that no number between $-4/p$ and p belongs to E and each element of E is real, then E is a type (B) convergence set (see, for example, [3, Corollary 2.2]). It is also true that every such set E is a type (B) convergence set with $V_B(E)$ bounded.

4. Let P denote the set of all complex numbers z such that $|z| - \text{Re}(z) < \frac{1}{2}$. By the Parabola Theorem of Scott and Wall [1] (Theorem 14.2 of [4]), if E is a bounded subset of P , then E is a type (A) convergence set. While Theorem A does not yield this result, it can be used to obtain a somewhat weaker result. Namely, if E is a bounded subset of P which is bounded away from the parabola, then it is a type (A) convergence set. It is quite easy to show that for such a set E , $V_A(E)$ is a bounded subset of the half-plane $\text{Re}(z) < -\frac{1}{2}$ and that it is bounded away from the line $\text{Re}(z) = -\frac{1}{2}$. Similar results can be obtained for certain "rotated" parabolic discs considered by Thron [2].

However, his General Parabola Theorem of that paper yields sharper results.

Let c be a real number and let C be the set of all complex numbers z such that $|z - c| \leq (c^2 + 4)^{1/2}$. In this case $V_B(C)$ is the set of all complex numbers z such that $|z + \frac{1}{2}c| < \frac{1}{2}(c^2 + 4)^{1/2}$. If E is a subset of C which is bounded away from the boundary of C (or, even bounded away from the common part of the boundary of C and the half-plane $\operatorname{Re}(z) \leq 0$), -1 is not a limit point of $V_B(E) \cdot V_B(E)$. Thus, by Theorem B, such a set E is a type (B) convergence set. However, C itself is a type (B) convergence set by Corollary 2.2 of [2].

In view of these examples and the results of §3, one might conjecture that the boundedness of the value region alone is necessary and sufficient for a set of complex numbers to be a convergence set. Unfortunately, we are unable to settle this question at this time.

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