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AN EXAMPLE OF A WEIGHT WITH TYPE III CENTRALIZER

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ABSTRACT. We give an example of a weight ϕ on the hyperfinite type II_∞ factor R, such that the centralizer M_ϕ of ϕ is a type III von Neumann algebra.

Introduction. Let ϕ be a normal, semifinite, faithful weight on a von Neumann algebra M, and σ_t^ϕ the modular automorphism group associated with ϕ. The centralizer M_ϕ is the fixed point-algebra under σ_t^ϕ, i.e.,

$$M_\phi = \{ x \in M | \sigma_t^\phi(x) = x, t \in R \}.$$

The restriction of ϕ to M_ϕ is a trace. Hence if ϕ is *strictly semifinite* (i.e. the restriction of ϕ to M_ϕ is semifinite), then M_ϕ is a semifinite von Neumann algebra (cf. [1]). In particular, if ϕ is bounded, then M_ϕ is finite. Recently A. Connes and M. Takesaki have proved that M_ϕ is semifinite for a great class of non strictly semifinite weights, namely the *integrable* weights, i.e. those weights ϕ, for which the set $\{x \in M_+ | \int_{-\infty}^{\infty} \sigma_t^\phi(x)\, dt \in M_+\}$ is σ-weakly dense in M_+ (cf. [3]).

In this paper we prove that M_ϕ is not, in general, semifinite. At the same time we find that the relative commutant $N' \cap M$ of a semifinite von Neumann algebra N in a semifinite von Neumann algebra M is not, in general, semifinite (Lemma 2). That is the negative solution to a problem considered in [2].

LEMMA 1. *On a separable Hilbert space H, there exists a hyperfinite type* II_1 *factor M and an abelian von Neumann algebra $A \subseteq M'$, such that the von Neumann algebra $(M \cup A)''$ is of type* III.

PROOF. Let $B = \otimes_{n=1}^{\infty} F_n$ be the uniformly hyperfinite C^*-algebra obtained as tensor product of a sequence of type I_2-factors (cf. [5, Chapter 1, §23]). B has a unique normalized trace τ, and $\pi_\tau(B)''$ is the hyperfinite type II_1-factor. Let U_n be the unitary group in F_n, and put $G = \prod_{n=1}^{\infty} U_n$. Then G is a compact group with normalized Haar measure dg. We define an action of G as *-automorphisms on B in the following way: For $g \in G$, $g = (u_n)_{n \in \mathbf{N}}$, we put

$$\sigma_g = \bigotimes_{n=1}^{\infty} \mathrm{ad}(u_n).$$

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It is easy to check that $g \to \sigma_g$ is a homeomorphism, and that $(g, x) \to \sigma_g x$ is jointly continuous. Let ϕ be a state on B, and put

$$\omega(x) = \int_G \phi \cdot \sigma_g(x)\, dg.$$

Let ω_n (resp. τ_n) be the restriction of ω (resp. τ) to $F_1 \otimes F_2 \otimes \cdots \otimes F_n$. By definition ω_n is invariant under $\mathrm{ad}(u_1) \otimes \cdots \otimes \mathrm{ad}(u_n)$, $u_k \in U_k$, $k = 1, \ldots, n$. From this it follows that $\omega_n = \tau_n$ for any $n \in N$, and thus $\omega = \tau$. Hence for any state ϕ on B,

$$\tau(x) = \int_G \phi \cdot \sigma_g(x)\, dg.$$

In the following we let ϕ be a fixed type III state on B. On the C^*-tensor product $B \otimes C(G) = C(G, B)$ we regard the state

$$\tilde{\phi}(x(g)) = \int_G \phi \cdot \sigma_g(x(g))\, dg, \qquad x \in B \otimes C(G).$$

$\tilde{\phi}$ is a type III state, because

$$\tilde{\phi}(x) = (\phi \otimes dg)\tilde{\sigma}(x), \qquad x \in B \otimes C(G),$$

where $\tilde{\sigma}$ is the *-automorphism of $B \otimes C(G)$ given by $\tilde{\sigma}(x)(g) = \sigma_g x(g)$. Hence $N = \pi_{\tilde{\phi}}(B \otimes C(G))''$ is of type III. However, N is generated by $\pi_{\tilde{\pi}}(B \otimes I)''$ and $\pi_{\tilde{\phi}}(I \otimes C(G))''$. The first is isomorphic to the hyperfinite type II_1-factor $\pi_\tau(B)''$ because $\tilde{\phi}(x \otimes I) = \tau(x)$, $x \in B$, and the latter is commutative. Moreover the two algebras commute. Since $B \otimes C(G)$ is a separable C^*-algebra, the Hilbert space $H_{\tilde{\phi}}$ associated with $\pi_{\tilde{\phi}}$ is separable. This completes the proof.

LEMMA 2. *Let R be the hyperfinite factor of type* II_∞. *There exists an abelian sub von Neumann algebra $A \subseteq R$, such that the relative commutant $R \cap A'$ is of type* III.

PROOF. Let M and A be as in Lemma 1. Put $R = M'$. Then $R \cap A' = (M \cup A)'$ is of type III. Since R contains a type III von Neumann algebra, it is of infinite multiplicity. Hence $R \simeq M \otimes F_\infty$ where F_∞ is a type I_∞ factor.

THEOREM. *There exists a normal, semifinite faithful weight on the hyperfinite factor of type* II_∞, *such that the centralizer M_ϕ is of type* III.

PROOF. Let A and R be as in Lemma 2. Since A is an abelian von Neumann algebra, which can be represented faithfully on a separable Hilbert space, A is the von Neumann algebra generated by a single selfadjoint element h. We may assume that $h \geqslant 1$. Put $\phi = \tau(h\,\cdot\,)$ where τ is the trace on R. Then

$$\sigma_t^\phi(x) = h^{it} x h^{-it}, \qquad x \in R \quad (\text{cf. } [4]).$$

Hence $M_\phi = R \cap \{h\}' = R \cap A'$ which is of type III.

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