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A NOTE ON THE CONCORDANCE HOMOTOPY GROUP OF REAL PROJECTIVE SPACE

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ABSTRACT. By means of the mapping torus construction the following theorem is proved

THEOREM. *If $r \equiv 3 \bmod 4$ and $r \geqslant 7$, and $\mathcal{P}_r$ is a homotopy P_r, then there is an isomorphism* $\pi_0 \operatorname{Diff}^+: \mathcal{P}_r \cong \pi_0 \operatorname{Diff}^+: P_r$.

1. **Introduction.** For a manifold M, let $\operatorname{Diff}^+(M)$ be the group of diffeomorphisms of M isotopic to diffeomorphisms leaving some nonempty open set fixed. Let $\pi_0 \operatorname{Diff}^+: M$ denote that group factored by those concordant to the identity. Similarly, let $\operatorname{Diff}^+(M,A)$ denote the subgroup of $\operatorname{Diff}^+(M)$ of diffeomorphisms fixing A, and let $\pi_0 \operatorname{Diff}^+: (M,A)$ denote $\operatorname{Diff}^+(M,A)$ factored by the subgroup of those concordant $\bmod A$ to the identity.

Let P_r denote real projective space of dimension r. In [4], an author establishes an isomorphism $\pi_0 \operatorname{Diff}^+: P_r \cong \pi_{r+1+k}(P_\infty/P_k)$ for $r \equiv 11 \bmod 16$ and $k = \alpha 2^L - r - 1$ with α a positive integer and L a large positive integer. Suppose M is a smooth closed, $(l-1)$-connected and oriented manifold of dimension n with $l = [n/2]$. Suppose $\zeta: M \to M$ is a free smooth involution; then there is an equivariant embedding $(S^l, -1) \subset (M, \zeta)$ producing an embedding $P_l \subset M/\zeta$, and $\nu(M/\zeta)|P_l = k\eta_l$ where $\eta_l \in \widetilde{KO}(P_l)$ is the reduced canonical line bundle. The integer k is well defined $\bmod c(l)$, where $c(l)$ is the order of $KO(P_l)$, and its class $\bmod c(l)$ is called the *type* of ζ. For k and l even, let

$$I_{2l}(k) = \{(M,\zeta) \mid M \sim S^l \times S^l, \text{ type } \zeta = k\}/\approx,$$

where $\sim$ means homotopy equivalent, and $\approx$ means orientation-preserving equivariantly diffeomorphic. From [3] we recall that $I_{2l}(k)$ has a canonical group structure, and that for $l \equiv 6 \bmod 8$ and $k \equiv -2l \bmod c(l)$ we have an isomorphism $\pi_{2l+k}(P_\infty/P_{k-1}) \cong I_{2l}(k)$. Thus we have for such l and k the isomorphism $\pi_0 \operatorname{Diff}^+: P_{2l-1} \cong I_{2l}(k)$. We will say that a *homotopy* P_r is a smooth closed r-manifold $\mathcal{P}_r$ homotopy equivalent to P_r. It is the object of this note to find a generalization, for $\mathcal{P}_r$ a homotopy P_r with $r \equiv 3 \bmod 4$, of the isomorphism $\pi_0 \operatorname{Diff}: P_{2l-1} \equiv I_{2l}(k)$ above.

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In [1], the first author introduces abelian groups $I_n(k)$ generalizing the $I_{2l}(k)$ above. Suppose M is a smooth closed n-manifold homotopy equivalent to $S^{[n/2]} \times S^{[(n+1)/2]}$ and suppose $\zeta\colon M \to M$ is a smooth free involution. Let $l = [n/2]$. We will say (M, ζ) is *admissible* if there exist disjoint copies, $P, P' \subset M/\zeta$ of P_l such that $P' \subset M/\zeta - P$ is a homotopy equivalence. With n even, all free involutions are admissible, but when n is odd some are excluded. Then set

$$I_n(k) = \{(M, \zeta) \mid M \sim S^{[n/2]} \times S^{[(n+1)/2]}, \text{ type } \zeta = k, \zeta \text{ admissible}\}/\approx .$$

The equivalence relation $\approx$ is the same as before when k is even–orientation preserving equivariant diffeomorphism–but when k is odd, it is only equivariant diffeomorphism. From [1] we recall that $I_n(k)$ has a canonical abelian group structure, provided $n \geqslant 6$; also from [1] we recall that there is an exact sequence of abelian groups

$$\cdots \to \mathfrak{L}_{n+1}(Z_2, (-1)^k) \xrightarrow{\partial} I_n(k) \xrightarrow{p} \Omega_n(\lambda(l,k)) \xrightarrow{a} \mathfrak{L}_n(Z_2, (-1)^k),$$

where $\mathfrak{L}_j(Z_2, (-1)^k)$ is a certain quotient of the Wall surgery group $L_j(Z_2, (-1)^k)$, and $\Omega_n(\lambda(l,k))$ is a certain Lashof cobordism group. It follows, for example, that $I_n(k)$ is finitely generated.

Now we can state the main theorem of this note.

THEOREM 2. *If $r \equiv 3 \bmod 4$ and $r \geqslant 7$, and $\mathcal{P}_r$ is a homotopy P_r, then there is an isomorphism $\pi_0 \operatorname{Diff}^+ : \mathcal{P}_r \cong I_{r+1}(k)$ where $k \equiv -r - 1 \bmod c(l)$.*

COROLLARY. *If $\mathcal{P}_r$ is a homotopy P_r, $r \equiv 3 \bmod 4$ and $r \geqslant 7$, then $\pi_0 \operatorname{Diff}^+ : \mathcal{P}_r \cong \pi_0 \operatorname{Diff}^+ : P_r$.*

The theorem is an immediate consequence of the following theorem. If $\mathcal{P}_r$ is a homotopy P_r, there is an embedding, for $m = [r/2]$, unique up to isotopy $P_{m-1} \subset \mathcal{P}_r$ such that $\pi_1(P_{m-1}) \to \pi_1(\mathcal{P}_r)$ is an epimorphism. Let N_r be a tubular neighborhood of P_{m-1} in $\mathcal{P}_r$. Let $f\colon \pi_0 \operatorname{Diff}^+ : (\mathcal{P}_r, N_r) \to \pi_0 \operatorname{Diff}^+ : \mathcal{P}_r$ be the forgetful homomorphism. Then we have the following:

THEOREM 1. *Let $r \geqslant 5$ and let $\mathcal{P}_r$ be a homotopy P_r. Then there is a homomorphism $\tau\colon \pi_0 \operatorname{Diff}^+ : (\mathcal{P}_r, N_r) \to I_{r+1}(k)$, where $k \equiv -r - 1 \bmod c(l)$, such that*:

(1) kernel $(\tau) \subset$ kernel (f),
(2) $\tau(\text{kernel } (f)) \subset \partial\mathfrak{L}_{r+2}(Z_2, (-1)^k)$,
(3) *τ is an epimorphism.*

We continue to use the notation implicit above: Given r, we set $l = [(r+1)/2]$, $m = [r/2]$, $c(l) = \text{order } \widetilde{KO}(P_l)$, $k = \text{class of } -r - 1 \bmod c(l)$, and η_s = canonical line bundle over P_s. If (M, N) is a smooth manifold pair, $\nu(N\colon M)$ denotes the normal bundle of N in M; $\tau(M)$ denotes the tangent bundle of M. If $\mathcal{P}_r$ is a homotopy P_r we have again the embedding $P_{m-1} \subset \mathcal{P}_r$ and its tubular neighborhood $N_r \subset \mathcal{P}_r$. Since $\mathcal{P}_r$ is necessarily tangentially homotopy equivalent to P_r and since $r - (m-1) = l + 1 > (m-1) + 1$, we have that N_r is a smooth embedding of the cell bundle associated with $(l+1)\eta_{m-1}$. There is an obvious homomorphism $\pi_0 \operatorname{Diff}^+ : (\mathcal{P}_r, N_r) \to^f \pi_0 \operatorname{Diff}^+ : \mathcal{P}_r$.

To see that f is an epimorphism we introduce a homomorphism d: $\pi_0 \operatorname{Diff}^+ : \mathcal{P}_r \to Z_2$ defined as follows: If $x \in \pi_0 \operatorname{Diff}^+ : \mathcal{P}_r$, we may choose a representative $\varphi: \mathcal{P}_r \to \mathcal{P}_r$ of x such that φ fixes P_1 where $P_1 \subset P_{m-1} \subset \mathcal{P}_r$. Then $d\varphi: \nu(P_1: \mathcal{P}_r) \to \nu(P_1: \mathcal{P}_r)$ represents a well-defined element $d(x) \in \widetilde{KO}^{-1}(P_1) = Z_2$, and $x \to d(x)$ is a homomorphism.

Proposition 1. *d: $\pi_0 \operatorname{Diff}^+ : \mathcal{P}_r \to Z_2$ is trivial.*

Proof. We are indebted for the proof to R. Z. Goldstein. As in the definition of d, let φ represent x, such that φ fixes P_1. Let $g \in H^1(S^1 : Z_2)$ and $g' \in H^1(P_1 : Z_2)$ be the nontrivial elements. Let $x(\varphi) \in H^1(S^1 \times P_1 : Z_2)$ be 0 if $d(x) = 0$ and $\mathrm{pr}_1^* g$ if $d(x) = 1$. Let $y = \mathrm{pr}_2^* g'$. Let $S^1 \times_\varphi \mathcal{P}_r$ be the mapping torus of φ. Then $S^1 \times P_1 \subset S^1 \times_\varphi \mathcal{P}_r$ and we have that the Stiefel-Whitney class

$$\omega(\nu(S^1 \times P_1 : S^1 \times_\varphi \mathcal{P}_r)) = (1 + x(\varphi))(1+y)^{r-1}.$$

On the other hand, φ is homotopic to the identity, so $S^1 \times_\varphi \mathcal{P}_r$ has the homotopy type of $S^1 \times P_r$ and $\omega(\tau(S^1 \times_\varphi \mathcal{P}_r) | S^1 \times P_1) = (1+y)^{r+1}$. Since $\tau(S^1 \times P_1)$ is trivial,

$$\omega(\tau(S^1 \times_\varphi \mathcal{P}_r) | S^1 \times P_1) = \omega(\nu(S^1 \times P_1 : S^1 \times_\varphi \mathcal{P}_r)),$$

and we get $(1 + x(\varphi))(1+y)^{r-1} = (1+y)^{r+1}$, from which $x(\varphi) = 0$ follows since $y^2 = 0$. The proposition is proved.

If $x \in \pi_0 \operatorname{Diff}^+ : \mathcal{P}_r$, then there is a representative φ that fixes $P_{m-1} \subset \mathcal{P}_r$. We would like to find a representative that fixes N_r. The representative φ at most twists N_r by an element $d'(\phi) \in \widetilde{KO}^{-1}(P_{m-1})$.

Proposition 2. *f: $\pi_0 \operatorname{Diff}^+ : (\mathcal{P}_r, N_r) \to \pi_0 \operatorname{Diff}^+ : \mathcal{P}_r$ is an epimorphism.*

Proof. The map $\widetilde{KO}^{-1}(P_{m-1}) \to \widetilde{KO}^{-1}(P_1)$ carries $d'(\varphi) \to d(x)$. This map is an isomorphism for $m \not\equiv 0 \bmod 4$, so we are done in that case by Proposition 1. If $m \equiv 0 \bmod 4$, then $\widetilde{KO}^{-1}(P_{m-1}) \equiv \widetilde{KO}^{-1}(P_1)$ is onto with infinite cyclic kernel. Then $d'(\varphi) \neq 0$ implies that $\nu(S^1 \times P_{m-1} : S^1 \times_\varphi \mathcal{P}_r)$ has a nontrivial rational Pontrjagin class in dimension m, which is impossible, and the proposition is proved.

Now we construct the homomorphism τ: $\pi_0 \operatorname{Diff}^+ : (\mathcal{P}_r, N_r) \to I_{r+1}(k)$ for $r \geqslant 5$. Briefly, it is the mapping torus construction followed by 'surgery' of $S^1 \times N_r \cup S^1_+ \times \mathcal{P}_r$. We construct a smooth manifold triad $(X; \partial_0 X, \partial_1 X)$ such that $\partial X = \partial_0 X \cup \partial_1 X$ and $\partial\partial_0 X = \partial\partial_1 X = \partial_0 X \cap \partial_1 X$ as follows: $X = D^2 \times \mathcal{P}_r$. With S^1_+ and S^1_- the right and left hemicircles, respectively, we set $\Gamma = \text{closure}(\mathcal{P}_r - N_r)$, and $\partial_0 X = S^1_- \times \Gamma$, and $\partial_1 X = S^1 \times N_r \cup S^1_+ \times P_r$.

Now, if $x \in \pi_0 \operatorname{Diff}^+ : (\mathcal{P}_r, N_r)$ is represented by φ, the mapping torus $S^1 \times_\varphi \mathcal{P}_r$ contains a codimension 0 submanifold canonically isomorphic to $S^1 \times N_r \cup S^1_+ \times \mathcal{P}_r$. Thus we may construct a well-defined ' surgery' with $(X; \partial_0 X, \partial_1 X)$ in place of the usual $(D^{r+1}; S^l \times D^m, D^{l+1} \times S^{m-1})$: Set $Y = ((S^1 \times_\varphi \mathcal{P}_r) \times [0,1]) \cup X$ where $\partial_1 X$ is identified with $(S^1 \times N_r \cup S^1_+ \times \mathcal{P}_r) \times 1$ by means of the canonical diffeomorphism. Then $\partial Y = \partial_0 Y \coprod \partial_1 Y$ with $\partial_0 Y = (S^1 \times_\varphi \mathcal{P}_r) \times 0$ and $\partial_1 Y$ the other component of ∂Y. It is routine to check that $\partial_1 Y$ is the orbit manifold of a representative of an element $\tau(\varphi) \in I_{r+1}(k)$. This element $\tau(\varphi)$ is well defined. If φ' is another representative of x, there is a concordance fixed on N_r from φ to φ'. Constructing Y' for φ' as above, and a similar manifold for the concordance, we obtain an h-cobordism finally from $\partial_1 Y$ to $\partial_1 Y'$ so that $\tau(\varphi') = \tau(\varphi')$. Thus the map $\tau: \pi_0 \operatorname{Diff}^+ : (\mathcal{P}_r, N_r) \to I_{r+1}(k)$ is well defined by $\tau(x) = \tau(\varphi)$ for φ a representative of x.

To see that τ is a homomorphism, we describe τ a different way. Recall from [1] that the orbit space Q of an element of $I_{r+1}(k)$ is obtained by gluing two copies of $E(m\eta_l + 1)$ (the cell bundle associated with $m\eta_l + 1$) by means of a diffeomorphism $\varphi': \partial E(m\eta_l + 1) \to \partial E(m\eta_l + 1)$. Since an $(m+1)$-plane bundle over P_l admitting a nonzero section and stably equivalent to $m\eta_l$ is uniquely determined up to bundle equivalence, we have a diffeomorphism $\Gamma \times 0 \cup \Gamma \times 1 \cong \partial E(m\eta_l + 1)$ where $\partial\Gamma \times 0$ is glued to $\partial\Gamma \times 1$ by the identity. We have obvious homomorphisms

$$\pi_0 \operatorname{Diff}^+ : (\mathcal{P}_r, N_r) \to \pi_0 \operatorname{Diff}^+ : (\Gamma, \partial\Gamma) \to \pi_0 \operatorname{Diff}^+ : (\Gamma \times 0 \cup \Gamma \times 1, \Gamma \times 1)$$
$$\to \pi_0 \operatorname{Diff}^+ : E(m\eta_l + 1).$$

If x is represented by φ, and $\varphi \to \varphi'$ under the above composition, it is straightforward to check that $E(m\eta_l + 1) \times 0 \cup_{\varphi'} E(m\eta_l + 1) \times 1$ represents $\tau(x)$. Thus we have the commutative diagram:

$$\begin{tikzcd}
\pi_0 \operatorname{Diff}^+ : (\mathcal{P}_r, N_r) \arrow[r] \arrow[dr, "\tau"'] & \pi_0 \operatorname{Diff} : \partial E(m\eta_l + 1) \arrow[d] \\
& I_{r+1}(k)
\end{tikzcd}$$

But the horizontal map is already a homomorphism, and according to [1] the vertical map is a homomorphism onto. It follows that τ is a homomorphism.

PROPOSITION 3. $\operatorname{kernel} \tau \subset \operatorname{kernel} f$.

PROOF. Suppose $\tau(x) = 0$ with φ a representative of x. Then $\tau(\varphi)$ has orbit space $\partial E(m\eta_l + 2) = \partial E$. Then we have $S^1 \times_\varphi \mathcal{P}_r = \partial(Y \cup E)$, where E is glued to Y along $\partial Y_1 = \partial E$. We have $S^1_+ \times \mathcal{P}_r \subset \partial(Y \cup E) \subset Y \cup E$, and this composition of inclusions is a homotopy equivalence. Using an embedding $S^1_+ \times \mathcal{P}_r \times [0,1] \subset Y \cup E$ given by a boundary collar, an easy application of the relative h-cobordism theorem, as in [4], shows that there is a diffeomorphism $(S^1 \times_\varphi \mathcal{P}_r, 1 \times \mathcal{P}_r) \cong (S^1 \times \mathcal{P}_r, 1 \times \mathcal{P}_r)$, which is the identity on the relative part. It follows that $f(x) = 0$, and Proposition 3 is proved.

PROPOSITION 4. *τ is an epimorphism.*

PROOF. Suppose $z \in I_{r+1}(k)$ has orbit space Q. We know $Q = E(m\eta_l + 1) \cup_{\varphi'} E(m\eta_l + 1)$ for some diffeomorphism φ': $\partial E(m\eta_l + 1) \to \partial E(m\eta_l + 1)$; also $S^1_+ \times \Gamma = \partial_0 X$ is diffeomorphic to $E(m\eta_l + 1)$. Thus, the triad $(X; \partial_0 X, \partial_1 X)$ determines a surgery Y from $\partial_1 Y = Q$ to $\partial_0 Y$. It is routine to check that $\partial_0 Y \cong S^1 \times_\varphi \mathcal{P}_r$ (e.g. as in [4]). By Proposition 2, we may take $\varphi \in \pi_0$ Diff: $(\mathcal{P}_r, N_r)$, and clearly $\tau(\varphi) = y$. Proposition 4 is proved.

PROOF OF THEOREM 2. We need only to check that

$$\tau(\text{kernel} f) \subset \partial \mathcal{L}_{r+2}(Z_2, (-1)^k).$$

Recall the definition of the Lashof cobordism group appearing in the exact sequence of [1]. First, $P[l,k] \to^{\lambda(l,k)} BO$ is a fibration such that $P_\infty \to P[l,k] \to^{\lambda(l,k)} BO$ is the lth Moore-Postnikov factorization of a map $P_\infty \to BO$ classifying $k\eta_\infty$. Then $\Omega_{r+1}(\lambda(l,k))$ is the $(r+1)$st Lashof cobordism group defined by the fibration $\lambda(l,k)$. The map p: $I_{r+1}(k) \to \Omega_{r+1}(\lambda(l,k))$ is defined as follows: If $z \in I_{r+1}(k)$ has orbit space Q, and $P \subset Q$ is one of the canonical embeddings of P_l in Q, then there is a commutative diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d, phantom, "\cap"] & P_\infty \arrow[d] \\
Q \arrow[dr] & P[l,k] \arrow[d, "\lambda(l,k)"] \\
& BO
\end{tikzcd}$$

with $Q \to BO$ a Gauss map. The obstructions are zero to finding a unique lift mod P_∞ of Q to $P[l,k]$. This lift represents an element of $\Omega_{r+1}(\lambda(l,k))$ which is well defined to be $p(z)$.

Let $x \in$ kernel (f) have representative φ and let $\tau(x) = z \in I_{r+1}(k)$, and let Y be the cobordism from $S^1 \times_\varphi \mathcal{P}_r$ to Q, the orbit space of z. Since $x \in$ kernel$f$, we have a diffeomorphism $S^1 \times_\varphi \mathcal{P}_r \cong S^1 \times \mathcal{P}_r = \partial(D^2 \times \mathcal{P}_r)$. Gluing $D^2 \times \mathcal{P}_r$ to Y via this diffeomorphism we obtain a manifold Λ, which may be written $\Lambda = (D^2 \times \mathcal{P}_r) \times 0 \cup_\alpha (D^2 \times \mathcal{P}_r) \times 1$ with gluing map an embedding α: $(S^1 \times N_r \cup S^1_+ \times \mathcal{P}_r) \times 1 \subset (S^1 \times \mathcal{P}_r) \times 0$ such that $\alpha((t,\xi),1) = ((t,\xi),0)$ for $t \in S^1_+$. We consider the lifting problem set by the following diagram:

$$
\begin{tikzcd}
0 \times \mathcal{P}_r \times 0 \arrow[rr] \arrow[d, phantom, "\cap" description] & & P_\infty \arrow[d] \\
\Lambda \arrow[rr, dashed] \arrow[drr, "\text{Gauss}"'] & & P[l, k] \arrow[d] \\
& & BO
\end{tikzcd}
$$

It can be solved iff the lifting problem set by the following diagram can be solved:

$$
\begin{tikzcd}
(S^1 \times N_r \cup S^1_+ \times \mathcal{P}_r) \times 1 \arrow[rr, "\beta"] \arrow[d, phantom, "\cap" description] & & P_\infty \arrow[d] \\
(D^2 \times \mathcal{P}_r) \times 1 \arrow[rr, dashed] \arrow[drr, "\text{Gauss}"'] & & P[l, k] \arrow[d] \\
& & BO
\end{tikzcd}
$$

where $(S^1 \times N_r \cup S^1_+ \times \mathcal{P}_r) \times 1 \to^\beta P_\infty$ is α followed successively by projection $(S^1 \times \mathcal{P}_r) \times 0 \to \mathcal{P}_r$ and then $\mathcal{P}_r \to P_\infty$. But $\varphi \in \mathrm{Diff}^+ \mathcal{P}_r$ implies φ homotopic to the identity so there is a homotopy commutative diagram:

$$
\begin{tikzcd}
(S^1 \times N_r \cup S^1_+ \times \mathcal{P}_r) \times 1 \arrow[r, "\beta"] \arrow[d, phantom, "\cap" description] & P_\infty \\
(D^2 \times \mathcal{P}_r) \times 1 \arrow[ur] &
\end{tikzcd}
$$

From this diagram follows the solution of the second lifting problem, and so of the first. Let v: $\Lambda \to P[l, k]$ be that solution. Then $v|Q$ represents $p(z)$, and thus $0 = p(z) = p(\tau(x))$. From the exact sequence of [1], it follows that $\tau(x) \in \partial \mathfrak{L}_{r+2}(Z_2, (-1)^k)$, the proof of Theorem 2 is complete.

Theorem 1 is an immediate consequence of Theorem 2 and the fact that $\mathfrak{L}_s(z_2, +1) = 0$ for $s \equiv 1 \bmod 4$ [2].

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