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REGULARITY OF SOLUTIONS TO AN ABSTRACT INHOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

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ABSTRACT. Let $T(t)$, $t \geq 0$, be a strongly continuous semigroup of linear operators on a Banach space X with infinitesimal generator A satisfying $T(t)X \subset D(A)$ for all $t > 0$. Let f be a function from $[0, \infty)$ to X of strong bounded variation. It is proved that $u(t) =^{\text{def}} T(t)x + \int^{t}_{0} T(t-s)f(s)ds$, $x \in X$, is strongly differentiable and satisfies $du(t)/dt = Au(t) + f(t)$ for all but a countable number of $t > 0$.

1. **Introduction**. Let $T(t)$, $t \geqslant 0$, be a strongly continuous semigroup of bounded linear operators on the Banach space X with infinitesimal generator A and let f be an X-valued function on $[0, \infty)$. Our objective is to establish sufficient conditions so that the function

$$u(t) \overset{\text{def}}{=} T(t)x + \int_0^t T(t-s)f(s)\,ds, \qquad x \in X, \tag{1.1}$$

is a strong solution of the inhomogeneous linear differential equation

$$du(t)/dt = Au(t) + f(t), \qquad u(0) = x. \tag{1.2}$$

It is well known that $u(t)$ satisfies (1.2) for $t \geqslant 0$ provided that $x \in D(A)$ and f is continuously differentiable (see [4, Theorem 1.19, p. 486] or [5, Theorem 6.5, p. 135]). It is also well known that $u(t)$ satisfies (1.2) for $t > 0$ provided that $x \in X$, $T(t)$, $t \geqslant 0$, is homomorphic, and f is Hölder continuous (see [4, Theorem 1.27, p. 491] or [5, Theorem 6.7, p. 138]). The theorem which we will prove demonstrates that $u(t)$ satisfies (1.2) under the assumptions that $T(t)X \subset D(A)$ for $t > 0$ and f is of strong bounded variation. The main idea of our proof is to show that under our assumptions the integral in (1.1) lies in $D(A)$ and the image of this integral under A may be represented as a Stieltjes integral.

THEOREM. *Suppose $T(t)X \subset D(A)$ for all $t > 0$ and f is of strong bounded variation on* $[0, r]$. *For a given $x \in X$ let $u(t)$ be defined on* $[0, r]$ *by* (1.1). *Then, $u(t)$ satisfies the following*:

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(1.3) *$u(t) \in D(A)$ for $t \in (0, r]$ and $Au(t)$ is continuous on* $(0, r]$;

(1.4) $d^+u(t)/dt = Au(t) + f(t+)$ *for all* $t \in (0, r)$ *and* $d^+u(t)/dt$ *is continuous from the right on* $(0, r)$;

(1.5) $d^-u(t)/dt = Au(t) + f(t-)$ *for all* $t \in (0, r]$ *and* $d^-u(t)/dt$ *is continuous from the left on* $(0, r]$;

(1.6) $du(t)/dt = Au(t) + f(t)$ *for all but a countable number of points in* $[0, r]$ *and* $du(t)/dt$ *is continuous at all but a countable number of points in* $[0, r]$.

Before proving our theorem we first state some facts about Banach space-valued functions of strong bounded variation.

2. **Vector-valued functions of strong bounded variation**. Suppose f is of strong bounded variation from $[0, r]$ to X (according to the definition of [3, p. 59]). The following properties of f may be proved analogously to the case of real-valued functions of bounded variation (for a discussion of real-valued functions of bounded variation the reader is referred to [9, Chapter 2] or [2, Chapter II]):

(2.1) f has a right limit at each $t \in [0, r)$, denoted by $f(t+)$, and $f(\cdot+)$ is right continuous on $[0, r)$;

(2.2) f has a left limit at each $t \in (0, r]$, denoted by $f(t-)$, and $f(\cdot-)$ is left continuous on $(0, r]$;

(2.3) $f(\cdot-)$ is of strong bounded variation on $[0, r]$ (where for convenience we define $f(0-) = f(0)$), and if we define $\nu(t)$ to be the total variation of $f(\cdot-)$ between 0 and t, then ν is nondecreasing and left continuous on $(0, r]$;

(2.4) f is bounded on $[0, r]$ and continuous at all but a countable number of points in $[0, r]$.

3. **Proof of the theorem**. We first prove the lemmas below, each of which is under the hypothesis of the theorem. In what follows we will suppose that M

is a constant such that $|T(t)| \leqslant M$ for $0 \leqslant t \leqslant r$ (see [4, p. 484]) and ν is defined as in (2.3).

LEMMA 3.1. *If* $0 < t \leqslant r$, *then*

$$\int_0^t T(t-s)f(s)\,ds \in D(A); \tag{3.1}$$
$$A\int_0^t T(t-s)f(s)\,ds = \int_0^t dT(t-s)f(s-).$$

PROOF. Let $0 < t \leqslant r$. We observe that $\int_0^t T(t-s)f(s)\,ds$, the Riemann integral, exists since the integrand is bounded and continuous almost everywhere by virtue of (2.1) and the continuity properties of $T(t)$, $t \geqslant 0$. The function T from $[0, t]$ to $B(X, X)$ (where $B(X, X)$ denotes the Banach space of bounded linear operators on X) is bounded on $[0, t]$. Further, since $T(t)X \subset D(A)$ for $t > 0$, T is continuous from $(0, t]$ to $B(X, X)$ (see [3, Theorem 10.3.5, p. 310]). By (2.3) the set of discontinuities of $T(t-s)$, considered as a function of s in $[0, t]$ to $B(X, X)$, has ν measure 0. That is, $s \to T(t-s)$ is discontinuous only at t and, by (2.3), $\lim_{s\to t^-}\nu(s) = \nu(t)$. Thus, the Riemann-Stieltjes integral $\int_0^t dT(t-s)f(s-)$ exists in the sense that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{s_i\}_{i=0}^n$ is a chain from 0 to t such that $\sup_{i=1,\dots,n}|s_i - s_{i-1}| < \delta$, and $s_{i-1} \leqslant s_i' \leqslant s_i$, then

$$\left\| \sum_{i=1}^n (T(t-s_i) - T(t-s_{i-1}))f(s_i' -) - \int_0^t dT(t-s)f(s-) \right\| < \varepsilon \tag{3.2}$$

(see [2, Theorem 13.16, p. 65 and Theorem 11.7, p. 53]).

For each positive integer n let $s_i^n = it/n$, where $i = 0, 1, \dots, n$. Define g_n: $[0, t] \to X$ by $g_n(s) = T(t-s)f(s_i^n -)$, where $s_{i-1}^n < s \leqslant s_i^n$, $i = 1, \dots, n$, and $g_n(0) = T(t)f(0)$. By (2.4), $\{g_n\}$ is bounded on $[0, t]$ and $\{g_n\}$ converges to $T(t-s)f(s)$ almost everywhere on $[0, t]$. By the Lebesgue theorem,

$$\lim_{n\to\infty}\int_0^t g_n(s)ds = \lim_{n\to\infty}\sum_{i=1}^n \int_{s_{i-1}^n}^{s_i^n} T(t-s)f(s_i^n -)ds = \int_0^t T(t-s)f(s)ds \tag{3.3}$$

(see [3, Theorem 3.7.9, p. 83]). From [4, p. 486], $\int_0^t g_n(s)ds \in D(A)$ and

$$A\int_0^t g_n(s)ds = \sum_{i=1}^n (T(t-s_i^n) - T(t-s_{i-1}^n))f(s_i^n -). \tag{3.4}$$

Then, by (3.2), (3.3), (3.4), and the closedness of A we obtain (3.1).

LEMMA 3.2. $A\int_0^t T(t-s)f(s)ds$ *is continuous from the right in* t *on* $[0, r)$.

PROOF. Let $0 \leqslant t < r$. First, we show that

$$\lim_{h\to 0^+} A\int_t^{t+h} T(t+h-s)f(s)ds = 0. \tag{3.5}$$

We observe that an argument similar to that of Lemma 3.1 shows that

$$\int_t^{t+h} T(t+h-s)f(s)ds \in D(A)$$

and

$$A\int_t^{t+h} T(t+h-s)f(s)ds = \int_t^{t+h} dT(t+h-s)f(s-).$$

Take $h > 0$ and sufficiently small. If $\varepsilon > 0$ there is a chain $\{s_i\}_{i=0}^n$ from t to $t+h$ such that

$$\left\|\int_t^{t+h}(T(h) - T(t+h-s))\,df(s-)\right\|$$

$$\begin{aligned} &< \left\|\sum_{i=1}^{n}(T(h) - T(t+h-s_{i-1}))(f(s_i-) - f(s_{i-1}-))\right\| + \varepsilon \\ (3.6) \qquad &< 2M\sum_{i=2}^{n}\|f(s_i-) - f(s_{i-1}-)\| + \varepsilon \\ &< 2M(\nu(t+h) - \nu(s_1)) + \varepsilon. \end{aligned}$$

Then, (3.6) yields

$$\begin{aligned} &\left\|\int_t^{t+h}(T(h) - T(t+h-s))df(s-)\right\| \\ (3.7) \qquad &\leqslant 2M\Big(\nu(t+h) - \lim_{s\to t^+}\nu(s)\Big). \end{aligned}$$

An integration by parts (see [2, Theorem 11.7, p. 53]) together with (3.7) yields

$$\begin{aligned} \left\|A\int_t^{t+h} T(t+h-s)f(s)ds\right\| &= \left\|\int_t^{t+h} dT(t+h-s)f(s-)\right\| \\ &= \left\|-\int_t^{t+h} T(t+h-s)\,df(s-) + f((t+h)-) - T(h)f(t-)\right\| \\ (3.8) \qquad &= \left\|\int_t^{t+h}(T(h) - T(t+h-s))\,df(s-)\right. \\ &\qquad\qquad \left. + f((t+h)-) - T(h)f((t+h)-)\right\| \\ &< 2M\Big(\nu(t+h) - \lim_{s\to t^+}\nu(s)\Big) + \|(I - T(h))f((t+h)-)\|. \end{aligned}$$

In order to establish (3.5) we need only show that

$$(3.9) \qquad \lim_{h\to 0^+}\|(I - T(h))f((t+h)-)\| = 0.$$

But (3.9) holds by virtue of the fact that the range of $f(\cdot -)$ on $[0, r]$ lies in a compact set of X and $\lim_{h\to 0^+}(I - T(h))z = 0$ uniformly for z in a compact set. The right continuity of $A\int_0^t T(t-s)f(s)ds$ in t now follows from (3.5) and the fact that

$$A\int_0^{t+h} T(t+h-s)f(s)ds - A\int_0^t T(t-s)f(s)ds$$
$$= (T(h)-I)A\int_0^t T(t-s)f(s)ds + A\int_t^{t+h} T(t+h-s)f(s)ds.$$

LEMMA 3.3. *$A\int_0^t T(t-s)f(s)ds$ is continuous from the left in t on* $(0, r]$.

PROOF. Let $0 < t \leqslant r$. Observe that for $c > 0$ and sufficiently small,

$$\begin{aligned} \left\| A\int_{t-c}^t T(t-s)f(s)ds \right\| &= \left\| \int_{t-c}^t dT(t-s)f(s-) \right\| \\ &= \left\| -\int_{t-c}^t T(t-s)df(s-) + f(t-) - T(c)f((t-c)-) \right\| \\ &\leqslant M(\nu(t)-\nu(t-c)) + M\|f(t-)-f((t-c)-)\| \\ &\quad + \|(I-T(c))f(t-)\|. \end{aligned} \tag{3.10}$$

If $h > 0$ and $c > 0$ are both sufficiently small, then (3.10) applied twice below yields

$$\begin{aligned} &\left\| A\int_0^t T(t-s)f(s)ds - A\int_0^{t-h} T(t-h-s)f(s)ds \right\| \\ &= \Bigg\| AT(c)\bigg(\int_0^{t-h-c} (T(h)-I)T(t-h-c-s)f(s)ds \\ &\qquad\qquad + \int_{t-h-c}^{t-c} T(t-c-s)f(s)ds \bigg) \\ &\qquad + A\int_{t-c}^t T(t-s)f(s)ds - A\int_{t-h-c}^{t-h} T(t-h-s)f(s)ds \Bigg\| \\ &\leqslant |AT(c)| \bigg(\left\| \int_0^{t-h-c} (T(h)-I)T(t-h-c-s)f(s)ds \right\| \\ &\qquad\qquad + Mh \sup_{s\in[t-h-c,t-c]} \|f(s)\| \bigg) \\ &\quad + M(\nu(t)-\nu(t-c)) + M\|f(t-)-f((t-c)-)\| \\ &\quad + \|(I-T(c))f(t-)\| + M(\nu(t-h)-\nu(t-h-c)) \\ &\quad + M\|f((t-h)-)-f((t-h-c)-)\| \\ &\quad + \|(I-T(c))f((t-h)-)\|. \end{aligned} \tag{3.11}$$

For a given $\varepsilon > 0$ first choose $c > 0$ and then choose $\delta > 0$ such that if $0 < h < \delta$, then (3.11) is $< \varepsilon$ (use the fact that ν and $f(\cdot -)$ are left continuous at t and $\lim_{h\to 0^+}(T(h)-I)z = 0$ uniformly for z in a compact set). The left continuity of $A\int_0^t T(t-s)f(s)ds$ then follows immediately.

To complete the proof of the theorem we see that (1.3) follows from

Lemmas 3.1, 3.2, and 3.3. To prove (1.4) let $0 < t < r$ and observe that for $h > 0$ and sufficiently small we have

$$(u(t+h) - u(t))/h = (T(t+h)x - T(t)x)/h + \frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds + \frac{T(h)-I}{h}\int_0^t T(t-s)f(s)ds.$$

By (2.1)

$$\lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds = f(t+)$$

and (1.4) then follows from Lemmas 3.1 and 3.2. To prove (1.5) let $0 < t \leqslant r$ and observe that for $h > 0$ and sufficiently small we have

$$(u(t-h) - u(t))/(-h) = (T(t-h)x - T(t)x)/(-h) + \frac{1}{h}\int_{t-h}^{t} T(t-s)f(s)ds + \frac{T(h)-I}{h}\int_0^{t-h} T(t-h-s)f(s)ds.$$

By (2.2)

$$\lim_{h\to 0^+} \frac{1}{h}\int_{t-h}^{t} T(t-h-s)f(s)ds = f(t-).$$

Denote

$$z(h) \overset{\text{def}}{=} \int_0^{t-h} T(t-h-s)f(s)ds.$$

By Lemma 3.1, $z(h) \in D(A)$ and by Lemma 3.3,

$$\lim_{h\to 0^+} \frac{T(h)-I}{h} z(h) = \lim_{h\to 0^+} \frac{1}{h}\int_0^h T(s)Az(h)ds = Az(0),$$

which yields (1.5). Finally, (1.6) follows immediately from (1.4), (1.5), and (2.4).

We conclude with the observation that our theorem may be applied to nonlinear evolution equations of the form $du(t)/dt = Au(t) + B(u(t))$. If $-B$ is an accretive continuous everywhere defined nonlinear operator on X, then there exists a solution $u(t)$ to the Volterra integral equation

$$u(t) = T(t)x + \int_0^t T(t-s)B(u(s))ds, \qquad x \in X$$

(see [10, Theorem I]). If we assume that $x \in D(A)$, then it can be shown that $u(t)$ is Lipschitz continuous. If we also assume that B is Lipschitz continuous and $T(t)X \subset D(A)$ for all $t > 0$, then our theorem implies $u(t)$ satisfies $du(t)/dt = Au(t) + B(u(t))$ for all $t \geqslant 0$. If it is not true that $T(t)X \subset D(A)$, then this conclusion may not hold (see [10, Example 4.1]). A similar observation is made in [7] for the case that $T(t)$, $t \geqslant 0$, is a holomorphic semigroup.

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