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MULTIPLIERS ON DUAL A^*-ALGEBRAS

B. J. TOMIUK

ABSTRACT. Let A be an A^*-algebra which is a dense $*$-ideal of a B^*-algebra $\mathfrak{A}$. We use tensor products and the algebra $M_l(A)$ of left multipliers on A to obtain a characterization of duality in A. We show, moreover, that if A is dual then $M_l(A)$ is algebra isomorphic to the second conjugate space $\mathfrak{A}^{**}$ of $\mathfrak{A}$ when $\mathfrak{A}^{**}$ is given Arens product.

1. **Introduction.** Let A be an A^*-algebra which is a dense $*$-ideal of a B^*-algebra A. In [8] a necessaary and sufficient condition was given for A to be dual which was expressed in terms of the weak operator topology on $M_r(A)$, the algebra of right multipliers on A, and a certain property of A called property (P2). In this paper we give several characterizations of property (P2) and then use some of them to give conditions for duality in A. Our presentation here differs somewhat from that in [8]. We use the tensor product approach as developed in [3] and [6]. Particularly in §2 we follow closely the presentation given in [3].

We shall use the notation of [8]. An A^*-algebra A is said to be of the first kind if it is an ideal of its completion $\mathfrak{A}$ in the auxiliary norm $|\cdot|$. It follows that there exists a constant $k > 0$ such that $\|xy\| \leqslant k\|x\|\,|y|$ for all $x \in A, y \in \mathfrak{A}$ [4, Lemma 4, p. 18]. If A is a modular annihilator A^*-algebra then $|\cdot|$ is unique [1, (1.3), p. 6] so that $\mathfrak{A}$ is also unique.

2. **The property (P2).** Let A be a Banach algebra, A^* and A^{**} its first and second conjugate spaces. Let $A \hat{\otimes} A^*$ be the projective tensor product of A and A^* [7, pp. 92–95]. Then $A \hat{\otimes} A^*$ is a Banach space with elements of the form $\sum_{k=1}^{\infty} a_k \hat{\otimes} f_k$ such that $\sum_{k=1}^{\infty}\|a_k\|\,\|f_k\| < \infty$, $a_k \in A, f_k \in A^*$, and the norm given by

$$\|h\| = \inf\left\{ \sum_{k=1}^{\infty} \|a_k\|\,\|f_k\| : h = \sum_{k=1}^{\infty} a_k \hat{\otimes} f_k \right\}.$$

For $a \in A, f \in A^*$, let $af \in A^*$ be given by $(af)x = f(xa)$, $x \in A$. This makes A^* into a left Banach A-module. We note that if $\sum_{k=1}^{\infty} a_k \hat{\otimes} f_k \in A \hat{\otimes} A^*$, then $\sum_{k=1}^{\infty} a_k f_k \in A^*$. Let ψ be the continuous linear map of $A \hat{\otimes} A^*$ into A^* given by

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$$\psi(a \hat{\otimes} f) = af \qquad (a \in A, f \in A^*).$$

For $a \in A$, $F \in A^{**}$, let $Fa \in A^{**}$ be given by $(Fa)f = F(af)$, $f \in A^*$. Then A^{**} is a right Banach A-module. (See [6] for the definition and properties of Banach A-modules.)

Let $A \circ A^*$ be the Banach space $A \hat{\otimes} A^*/\ker(\psi)$ with the usual quotient norm, where $\ker(\psi)$ is the kernel of ψ. Then $(A \circ A^*)^*$ consists of all those $\mathfrak{F} \in (A \hat{\otimes} A^*)^*$ which vanish on $\ker(\psi)$. Now, for each $F \in A^{**}$, let $\mathfrak{F}_F \in (A \hat{\otimes} A^*)^*$ be given by

$$\mathfrak{F}_F(a \hat{\otimes} f) = F\big(\psi(a \hat{\otimes} f)\big) = F(af) \qquad (a \in A, f \in A^*).$$

Each $\mathfrak{F}_F$ vanishes on $\ker(\psi)$, so that $\{\mathfrak{F}_F\colon F \in A^{**}\}$ may be indentified as a subspace of $(A \circ A^*)^*$. Moreover if $Fa = 0$ for all $a \in A$ implies $F = 0$, then $F \to \mathfrak{F}_F$ is a one-to-one map of A^{**} into $(A \hat{\otimes} A^*)^*$. Thus in this case $F \to \mathfrak{F}_F$ identifies A^{**} as a subspace of $(A \circ A^*)^*$.

Let σ denote the w^*-topology of $(A \hat{\otimes} A^*)^*$.

LEMMA 2.1. *$(A \circ A^*)^*$ is the σ-closure of* $\{\mathfrak{F}_F\colon F \in A^{**}\}$.

PROOF. We have

$$\ker(\psi) = \left\{ \sum_{k=1}^{\infty} a_k \hat{\otimes} f_k \in A \hat{\otimes} A^*\colon \sum_{k=1}^{\infty} a_k f_k = 0 \right\}$$

$$= \left\{ \sum_{k=1}^{\infty} a_k \hat{\otimes} f_k \in A \hat{\otimes} A^*\colon F\left(\sum_{k=1}^{\infty} a_k f_k \right) = 0,\ F \in A^{**} \right\}.$$

Thus $\ker(\psi) = \bigcap[\ker(\mathfrak{F}_F)\colon F \in A^{**}]$, which means that $\ker(\psi)$ is the polar of $\{\mathfrak{F}_F\colon F \in A^{**}\}$. Therefore, by the Bipolar Theorem [7, p. 126], $(A \circ A^*)^*$ is the σ-closure of $\{\mathfrak{F}_F\colon F \in A^{**}\}$. This completes the proof.

We observe that if $A^2 = (0)$, then $\ker(\mathfrak{F}_F) = A \hat{\otimes} A^*$, for every $F \in A^{**}$, so that $\ker(\psi) = A \hat{\otimes} A^*$ and consequently $(A \circ A^*)^* = (0)$.

Let $\mathfrak{B}(A, A^{**})$ be the Banach space of all bounded linear operators $T\colon A \to A^{**}$ normed with the operator bound norm. For each $\mathfrak{F} \in (A \hat{\otimes} A^*)^*$, let $T_{\mathfrak{F}}$ be the map on A into A^{**} given by

$$(f, T_{\mathfrak{F}}(a)) = \mathfrak{F}(a \hat{\otimes} f) \qquad (a \in A, f \in A^*).$$

Then clearly $T_{\mathfrak{F}} \in \mathfrak{B}(A, A^{**})$ for every $\mathfrak{F} \in (A \hat{\otimes} A^*)^*$, and it is easy to check that the map $\phi\colon \mathfrak{F} \to T_{\mathfrak{F}}$ is an isometric isomorphism of $(A \hat{\otimes} A^*)^*$ onto $\mathfrak{B}(A, A^{**})$. Give $\mathfrak{B}(A, A^{**})$ the image of the σ topology by the map ϕ. Following Máté [3], we shall refer to this topology as the ultraweak topology on $\mathfrak{B}(A, A^{**})$.

Now consider A^{**} as a right Banach A-module and let $\mathrm{Hom}_A(A, A^{**})$ be the set of all $T \in \mathfrak{B}(A, A^{**})$ such that $T(ab) = T(a)b$, $a, b \in A$. The canonical map $\pi\colon A \to A^{**}$ belongs to $\mathfrak{B}(A, A^{**})$ since $\pi(ab)f = f(ab) = \pi(a)(bf)$ for all $a, b \in A$ and $f \in A^*$. For each $F \in A^{**}$, let $T_F\colon A \to A^{**}$ be given by $T_F(a) = Fa$, $a \in A$. Then $T_F \in \mathrm{Hom}_A(A, A^{**})$, and we have $\phi(\mathfrak{F}_F)$

$= T_F$ for all $F \in A^{**}$. In view of Lemma 2.1 and the fact that the ultraweak closure of $\{T_F: F \in A^{**}\} \subseteq \mathrm{Hom}_A(A, A^{**})$ we have

LEMMA 2.2. $\phi((A \circ A^*)^*) \subseteq \mathrm{Hom}_A(A, A^{**})$ *and is the ultraweak closure of* $\{T_F: F \in A^{**}\}$.

If ϕ maps $(A \circ A^*)^*$ onto $\mathrm{Hom}_A(A, A^{**})$, we shall write $(A \circ A^*)^* \cong \mathrm{Hom}_A(A, A^{**})$. In this case, for every $T \in \mathrm{Hom}_A(A, A^{**})$, $\mathcal{F}_T \in (A \hat{\otimes} A^*)^*$, given by $\mathcal{F}_T(a \hat{\otimes} f) = (f, T(a))$, belongs to $(A \circ A^*)^*$. In particular,

$$\mathcal{F}_\pi(a \hat{\otimes} f) = f(a), \quad \text{for all } a \in a, f \in A^*.$$

We recall that a Banach algebra A is said to have property (P2) if: $a_k \in A$, $f_k \in A^*$, $\sum_{k=1}^\infty \|a_k\| \, \|f_k\| < \infty$ and $\sum_{k=1}^\infty a_k f_k = 0$ implies that $\sum_{k=1}^\infty f_k(a_k) = 0$. (This is the left-hand version of the definition given in [8].)

THEOREM 2.3. *Let A be a Banach algebra. Then the following statements are equivalent*:

(i) *A has property* $(P2)$.

(ii) *For* $h = \sum_{k=1}^\infty a_k \hat{\otimes} f_k \in \ker(\psi)$ *we have* $\sum_{k=1}^\infty f_k(a_k) = 0$.

(iii) $\mathcal{F}_\pi$ *vanishes on* $\ker(\psi)$.

(iv) $\mathcal{F}_\pi \in (A \circ A^*)^*$.

(v) $\mathrm{Hom}_A(A, A^{**}) \cong (A \circ A^*)^*$.

(vi) *There exists a net* $\{u_\alpha\}$ *in A such that* $\{\mathcal{F}_{\pi(u_\alpha)}\}$ *converges to* $\mathcal{F}_\pi$ *in the w^*-topology on* $(A \hat{\otimes} A^*)^*$.

PROOF. (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) are clear.

(iv) $\Rightarrow$ (vi). Suppose (iv) holds. Then $\ker(\mathcal{F}_\pi) \supset \ker(\psi)$. We have

$$\mathcal{F}_\pi(h) = \mathcal{F}_\pi\left(\sum_{k=1}^\infty a_k \hat{\otimes} f_k\right) = \sum_{k=1}^\infty \mathcal{F}_\pi(a_k \hat{\otimes} f_k) = \sum_{k=1}^\infty f_k(a_k),$$

for all $h = \sum_{k=1}^\infty a_k \hat{\otimes} f_k \in A \hat{\otimes} A^*$. Since $\{\mathcal{F}_F: F \in A^{**}\}$ is σ-dense in $(A \circ A^*)^*$, there exists a net $\{F_\alpha\}$ in A^{**} such that $\mathcal{F}_{F_\alpha}(h) \to \mathcal{F}_\pi(h)$ for all $h \in A \hat{\otimes} A^*$. Since $\pi(A)$ is w^*-dense in A^{**} and σ is weaker than the w^*-topology on A^{**}, it follows that $\{\mathcal{F}_{\pi(a)}: a \in A\}$ is σ-dense in $(A \circ A^*)^*$. Hence there exists a net $\{u_\alpha\}$ in A such that $\{\mathcal{F}_{\pi(u_\alpha)}\}$ σ-converges to $\mathcal{F}_\pi$. We have

$$\begin{aligned}\mathcal{F}_{\pi(u_\alpha)}(h) &= \pi(u_\alpha)(\psi(h)) = \sum_{k=1}^\infty \pi(u_\alpha)(a_k f_k) \\ &= \left(\sum_{k=1}^\infty a_k f_k\right)(u_\alpha) = \sum_{k=1}^\infty (a_k f_k)(u_\alpha) \\ &= \sum_{k=1}^\infty f_k(u_\alpha a_k).\end{aligned}$$

Thus

$$(1) \qquad \lim_{\alpha} \mathcal{F}_{\pi(u_\alpha)}(h) = \lim_{\alpha} \sum_{k=1}^{\infty} f_k(u_\alpha a_k) = \sum_{k=1}^{\infty} f_k(a_k) = \mathcal{F}_\pi(h),$$

for all $h = \sum_{k=1}^{\infty} a_k \hat{\otimes} f_k \in A \hat{\otimes} A^*$.

(vi) $\Rightarrow$ (v). We have $\phi((A \circ A^*)^*) \subseteq \mathrm{Hom}_A(A, A^{**})$. We need only show that $\mathrm{Hom}_A(A, A^{**}) \subseteq \phi((A \circ A^*)^*)$. Let $T \in \mathrm{Hom}_A(A, A^{**})$ and let $\mathcal{F}_T$ be the corresponding element of $(A \hat{\otimes} A^*)^*$. Then, using (1), we obtain (identifying A as a subset of A^{**} and A^* as a subset of A^{***}):

$$\begin{aligned} \mathcal{F}_T\left(\sum_{k=1}^{\infty} a_k \hat{\otimes} f_k\right) &= \sum_{k=1}^{\infty} \mathcal{F}_T(a_k \hat{\otimes} f_k) = \sum_{k=1}^{\infty} (f_k, T(a_k)) \\ &= \sum_{k=1}^{\infty} (T^* f_k)(a_k) = \lim_{\alpha} \sum_{k=1}^{\infty} (T^* f_k)(u_\alpha a_k) \\ &= \lim_{\alpha} \sum_{k=1}^{\infty} f_k(T(u_\alpha a_k)) = \lim_{\alpha} \sum_{k=1}^{\infty} f_k(T(u_\alpha) a_k) \\ &= \lim_{\alpha} \sum_{k=1}^{\infty} (T^* a_k f_k)(u_\alpha) = \sum_{k=1}^{\infty} T^*(a_k f_k), \end{aligned}$$

where T^* is the conjugate of T. Hence if $\sum_{k=1}^{\infty} a_k f_k = 0$ then $\mathcal{F}_T(\sum_{k=1}^{\infty} a_k \hat{\otimes} f_k) = 0$, so that $\ker(\psi) \subset \ker(\mathcal{F}_T)$. Thus $\mathrm{Hom}_A(A, A^{**}) \subseteq \phi((A \circ A^*)^*)$ and so $\mathrm{Hom}_A(A, A^{**}) \cong (A \circ A^*)^*$.

(v) $\Rightarrow$ (iv). This is clear since $\pi \in \mathrm{Hom}_A(A, A^{**})$.

3. **Dual A^*-algebras.** Let A be a Banach algebra. A map $T: A \to A$ is called a left (resp. right) multiplier if $T(ab) = T(a)b$ (resp. $T(ab) = aT(b)$), for all $a, b \in A$. Let $M_l(A)$ (resp. $M_r(A)$) be the set of all bounded linear left (resp. right) multipliers on A. $M_l(A)$ and $M_r(A)$ are Banach algebras under the usual operations for operators and the operator bound norm. We observe that if $T \in M_l(A)$ then the composite map $\pi \circ T \in \mathrm{Hom}_A(A, A^{**})$. Let ϕ_π be the map of $M_l(A)$ into $\mathrm{Hom}_A(A, A^{**})$ given by

$$\phi_\pi(T) = \pi \circ T \qquad (T \in M_l(A)).$$

For any Banach space X, let $\mathbb{S}(X)$ denote the closed unit ball of X. It follows from the proof of [8, Theorem 4.7, p. 286] that if A is a dual A^*-algebra of the first kind then $\mathbb{S}(M_l(A))$ is τ_l-compact, where τ_l is the weak operator topology on $M_l(A)$. (We take the left-hand version of the arguments in [8, p. 286].)

THEOREM 3.1. *Let A be an A^*-algebra of the first kind. Then the following statements are equivalent*:

(i) *A is dual.*

(ii) *$\phi_\pi(M_l(A))$ is the ultraweak closure of $\{T_F: F \in A^{**}\}$.*

PROOF. (i) $\Rightarrow$ (ii). Suppose A is dual. Then, by [8, Theorem 4.7, p. 286], it has property (P2) and therefore, by Theorem 2.3, $(A \circ A^*)^* \cong \mathrm{Hom}_A(A, A^{**})$. Hence $\mathrm{Hom}_A(A, A^{**})$ is the ultraweak closure of $\{T_F: F \in$

$A^{**}\}$. Now $T_F(a)f = (Fa)f = (F * \pi(a))f$ and, by [9, Theorem 5.2, p. 830], $\pi(A)$ is an ideal of A^{**} when A^{**} is given either Arens product, so that $T_F(a) \in \pi(A)$ for all $a \in A$. Therefore $T_F = \pi \circ T$, for some $T \in M_l(A)$, and so $\{T_F: F \in A^{**}\} \subseteq \phi_\pi(M_l(A))$. Let $Q \in \mathrm{Hom}_A(A, A^{**})$. Then, by Lemma 2.2 and the fact that $(A \circ A^*)^* \cong \mathrm{Hom}_A(A, A^{**})$, there exists a net $\{F_\alpha\}$ in A^{**} such that $(f, T_{F_\alpha}(a)) \to (f, Q(a))$ for all $a \in A, f \in A^*$. Let $T_\alpha \in M_l(A)$ be such that $T_{F_\alpha} = \pi \circ T_\alpha$, for all α. Then $(f, T_{F_\alpha}(a)) = f(T_\alpha(a))$ since $T_\alpha(a) \in A$. But, by [8, Theorem 4.7, p. 286], $M_l(A)$ is τ_l-complete. Hence there exists $T \in M_l(A)$ such that $f(T_\alpha(a)) \to f(T(a))$, for all $a \in A, f \in A^*$. This shows that $\pi(T(a))f = (f, Q(a))$, for all $a \in A, f \in A^*$, or equivalently, $\pi(T(a)) = Q(a)$, for all $a \in A$, i.e., $Q = \pi \circ T$. Thus $Q \in \phi_\pi(M_l(A))$ and so $\mathrm{Hom}_A(A, A^{**}) = \phi_\pi(M_l(A))$. Since $(A \circ A^*)^* \cong \mathrm{Hom}_A(A, A^{**})$ and since $(A \circ A^*)^*$ is the σ-closure of $\{\mathfrak{F}_F: F \in A^{**}\}$, it follows that $\phi_\pi(M_l(A))$ is the ultraweak closure of $\{T_F: F \in A^{**}\}$.

(ii) $\Rightarrow$ (i). Suppose (ii) holds. Then, in view of Lemma 2.2, $M_l(A)$ is isometrically isomorphic to $(A \circ A^*)^*$. From Lemma 2.1 we obtain $\mathfrak{S}((A \circ A^*)^*)$ is σ-compact, and since τ_l is weaker than the ultraweak topology on $M_l(A)$, it follows that $\mathfrak{S}(M_l(A))$ is τ_l-compact and therefore τ_l-complete. Let I be the identity element of $M_l(A)$. Since $\{T_F: F \in A^{**}\}$ is ultraweak dense in $\phi_\pi(M_l(A))$, there exists a net $\{F_\alpha\}$ in A^{**} such that T_{F_α} converges ultraweakly to $\pi \circ I = \pi$, or equivalently, $\mathfrak{F}_{F_\alpha}$ σ-converges to $\mathfrak{F}_\pi$. Since $\pi(A)$ is w^*-dense in A^{**} and the w^*-topology is stronger than the σ-topology on A^{**}, it follows that there exists a net $\{u_\alpha\}$ in A such that $\{\mathfrak{F}_{\pi(u_\alpha)}\}$ σ-converges to $\mathfrak{F}_\pi$ and so, by Theorem 2.3, A has property (P2). Therefore, by [8, Theorem 4.7, p. 287], A is dual.

COROLLARY 3.2. *Let A be an A^*-algebra of the first kind. Then A is dual if and only if* $\phi_\pi(M_l(A)) = \phi((A \circ A^*)^*)$.

COROLLARY 3.3. *Let A be a modular annihilator A^*-algebra of the first kind. If* $\mathrm{Hom}_A(A, A^{**})$ *is the ultraweak closure of* $\{T_F: F \in A^{**}\}$ *then A is dual.*

PROOF. By [9, Theorem 5.2, p. 830], $\pi(A)$ is an ideal of A^{**} so that T_F maps A into $\pi(A)$ for every $F \in A^{**}$. Hence if $\mathrm{Hom}_A(A, A^{**})$ is the ultraweak closure of $\{T_F: F \in A^{**}\}$, then $\mathrm{Hom}_A(A, A^{**}) = \phi_\pi(M_l(A))$ by the proof above. Therefore A is dual by Theorem 3.1.

4. A realization of the algebra $M_l(A)$.

THEOREM 4.1. *Let A be a dual A^*-algebra of the first kind and let $\mathfrak{A}$ be its completion. Let $\pi_{\mathfrak{A}}$ be the canonical map of $\mathfrak{A}$ into $\mathfrak{A}^{**}$. Then $\pi_{\mathfrak{A}}(A)$ is an ideal of $\mathfrak{A}^{**}$ when $\mathfrak{A}^{**}$ is given Arens product.*

PROOF. Let $x \in A$, $F \in \mathfrak{A}^{**}$ and let $\{e_\alpha\}$ be a maximal orthogonal family of selfadjoint minimal idempotents in A. By [4, Theorem 16, p. 30], $\Sigma_\alpha e_\alpha x$ is summable to x in the norm $\|\cdot\|$, and hence there exists only a countable number of e_α for which $e_\alpha x \neq 0$, say $e_{\alpha_1}, e_{\alpha_2}, \ldots$. Since A and $\mathfrak{A}$ have the same socle and $\pi_{\mathfrak{A}}(\mathfrak{A})$ is an ideal of $\mathfrak{A}^{**}$, it follows that $F * \pi_{\mathfrak{A}}(e_{\alpha_i}) \in \pi_{\mathfrak{A}}(A)$ for $i = 1, 2, \ldots$. Let m, n be positive integers, $m \leqslant n$. By [4, Lemma 4, p. 18], we have

$$\left\| \sum_{i=1}^{n} F * \pi_{\mathfrak{A}}(e_{\alpha_i})\pi_{\mathfrak{A}}(x) - \sum_{i=1}^{m} F * \pi_{\mathfrak{A}}(e_{\alpha_i}) * \pi_{\mathfrak{A}}(x) \right\|$$
$$= \left\| \left(\sum_{i=m+1}^{n} F * \pi_{\mathfrak{A}}(e_{\alpha_i}) \right) \left(\sum_{i=m+1}^{n} \pi_{\mathfrak{A}}(e_{\alpha_i}) * \pi_{\mathfrak{A}}(x) \right) \right\|$$
$$\leqslant k \left\| \left(\sum_{i=m+1}^{n} F * \pi_{\mathfrak{A}}(e_{\alpha_i}) \right) \left(\sum_{i=m+1}^{n} \pi_{\mathfrak{A}}(e_{\alpha_i}) * \pi_{\mathfrak{A}}(x) \right) \right\|$$
$$\leqslant k|F| \left\| \sum_{i=m+1}^{n} \pi_{\mathfrak{A}}(e_{\alpha_i}) \right\| \left\| \sum_{i=m+1}^{n} \pi_{\mathfrak{A}}(e_{\alpha_i}) * \pi_{\mathfrak{A}}(x) \right\|$$
$$\leqslant k|F| \left\| \sum_{i=m+1}^{n} \pi_{\mathfrak{A}}(e_{\alpha_i}) * \pi_{\mathfrak{A}}(x) \right\|,$$

where $|F|$ denotes the norm of F in $\mathfrak{A}^{**}$ and k is a positive constant. Thus $\{\sum_{i=1}^{n} F * \pi_{\mathfrak{A}}(e_{\alpha_i}) * \pi_{\mathfrak{A}}(x)\}$ is a Cauchy sequence in $\pi_{\mathfrak{A}}(A)$ with respect to the norm $\|\cdot\|$, and so there exists $z \in A$ such that $\pi_A(z) = \sum_{i=1}^{\infty} F * \pi_{\mathfrak{A}}(e_{\alpha_i}) * \pi_{\mathfrak{A}}(x)$. Since $\sum_{i=1}^{\infty} F * \pi_{\mathfrak{A}}(e_{\alpha_i}) * \pi_{\mathfrak{A}}(x)$ also converges to $\pi_{\mathfrak{A}}(z)$ and to $F * \pi_{\mathfrak{A}}(x)$ in the norm $|\cdot|$, we have $\pi_{\mathfrak{A}}(z) = F * \pi_{\mathfrak{A}}(x)$. Hence $F * \pi_{\mathfrak{A}}(x) \in \pi_{\mathfrak{A}}(A)$, for all $x \in A$ and $F \in \mathfrak{A}^{**}$. Similarly we can show that $\pi_{\mathfrak{A}}(x) * F \in \mathfrak{A}^{**}$, for all $x \in A$ and $F \in \mathfrak{A}^{**}$. Therefore $\pi_{\mathfrak{A}}(A)$ is an ideal of $\mathfrak{A}^{**}$.

THEOREM 4.2. *Let A be a dual A^*-algebra of the first kind and $\mathfrak{A}$ its completion. Then $M_l(A)$ is algebra isomorphic to $\mathfrak{A}^{**}$ when $\mathfrak{A}^{**}$ is given Arens product. This isomorphism is given by the following relation: For each $T \in M_l(A)$ there exists a unique $F_T \in \mathfrak{A}^{**}$ such that*

$$\pi_{\mathfrak{A}}(Tx) = F_T * \pi_{\mathfrak{A}}(x) \qquad (x \in A).$$

PROOF. For each $x \in A$, let $\|x\|'_A = \sup\{\|xy\|: \|y\| \leqslant 1, y \in A\}$. Then $\|\cdot\|'_A$ is a norm on A which is equivalent to $|\cdot|$ [4, Theorem 18, p. 31]. Hence if $T \in M_l(A)$ and $x \in A$, then

$$\|Tx\|'_A = \sup\{\|T(x)y\|: \|y\| \leqslant 1, y \in A\}$$
$$= \sup\{\|T(xy)\|: \|y\| \leqslant 1, y \in A\}$$
$$\leqslant \|T\| \sup\{\|xy\|: \|y\| \leqslant 1, y \in A\}$$
$$\leqslant k'\|T\|\,|x|,$$

where k' is a constant > 0. Thus $|Tx| \leqslant k''|x|$ for all $x \in A$ and some constant $k'' > 0$. Since A is dense in $\mathfrak{A}$, it follows that T has a unique bounded extension T' to $\mathfrak{A}$. Clearly $T' \in M_l(\mathfrak{A})$. By [2, Corollary 3.2, p. 509], there exists a unique $F_T \in \mathfrak{A}^{**}$ such that $\pi_{\mathfrak{A}}(Tx) = F_T * \pi_{\mathfrak{A}}(x)$ for all $x \in A$.

Since, by Theorem 4.1, $\pi_{\mathfrak{A}}(A)$ is an ideal of $\mathfrak{A}^{**}$, we have that $T \to F_T$ is an algebra isomorphism of $M_l(A)$ onto $\mathfrak{A}^{**}$.

References

1. B. A. Barnes, *Subalgebras of modular annihilator algebras*, Proc. Cambridge Philos. Soc. **66** (1969), 5–12. MR **41** #4236.

2. B. D. Malviya and B. J. Tomiuk, *Multiplier operators on B^*-algebras*, Proc. Amer. Math. Soc. **31** (1972), 505–510. MR **46** #4215.

3. L. Máté, *On representation of module-homomorphisms* (*multipliers*), Studia Sci. Math. Hungar. **8** (1973), 187–192. MR **51** #1286.

4. T. Ogasawara and K. Yoshinaga, *Weakly completely continuous Banach*-algebras*, J. Sci. Hiroshima Univ. Ser. A **18** (1954), 15–36. MR **16**, 1126.

5. C. E. Rickart, *General theory of Banach algebras*, Van Nostrand, Princeton, N.J., 1960. MR **22** #5903.

6. M. Rieffel, *Induced Banach representations of Banach algebras and locally compact groups*, J. Functional Analysis **1** (1967), 443–491. MR **36** #6544.

7. H. H. Schaefer, *Topological vector spaces*, Macmillan, New York, 1966. MR **33** #1689.

8. B. J. Tomiuk, *Multipliers and duality in A^*-algebras*, Proc. Amer. Math. Soc. **50** (1975), 281–288. MR **51** #8834.

9. Pak-ken Wong, *Modular annihilator A^*-algebras*, Pacific J. Math. **37** (1971), 825–834. MR **46** #4231.

Department of Mathematics, University of Ottawa, Ottawa, Ontario, Canada K1N 6N5