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CONFORMAL INVARIANTS OF SUBMANIFOLDS

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ABSTRACT. A local conformal invariant and a global conformal invariant of a submanifold immersed in a Euclidean space are derived.

1. **Introduction.** It is well known (Haantjes [3]) that every conformal mapping f on a Euclidean m-space E^m can be decomposed into a product of similarity transformations (i.e., Euclidean motions and homotheties) and inversions $\{\pi_i\}$. Let $x\colon M^n \to E^m$ be an n-dimensional submanifold immersed in E^m. For simplicity we shall write $x(M^n)$ as M^n. A quantity on M^n is a *conformal invariant* if it is invariant under the conformal mappings of E^m, for which the center of every inversion does not lie on M^n.

Let e be a unit normal vector of M^n at a point x. Then the first fundamental form of M^n at x and the second fundamental form of M^n at x with respect to e are respectively defined to be

$$\mathrm{I} = dx \cdot dx, \quad \mathrm{II}(e_x) = -dx \cdot de, \tag{1.1}$$

where dx and de are vector-valued linear forms on M^n, and the dot denotes the inner product of two vectors in E^m; actually the form I is the Riemannian metric on M^n induced by the immersion. The eigenvalues $h_1(e), \ldots, h_n(e)$ of $\mathrm{II}(e)$ relative to I are called the principal curvatures of M^n at the point x with respect to e, and the rth mean curvature of M^n at x with respect to e is defined to be the rth elementary symmetric function of $h_1(e), \ldots, h_n(e)$ divided by the number of terms, i.e.,

$$\binom{n}{r} H_r(e) = \sum_{i_1 \ldots i_r = 1}^{n} h_{i_1}(e) \ldots h_{i_r}(e), \qquad r = 1, \ldots, n, \tag{1.2}$$

where $\binom{n}{r}$ is the binomial coefficient.

Let B be the bundle of unit normal vectors of M^n, so that a point of B is a pair (x, e). Then B is a bundle of $(m - n - 1)$-dimensional spheres S^{m-n-1} of unit normal vectors over M^n and is a manifold of dimension $m - 1$. Let dV_n and $d\sigma_{m-n-1}$ be the volume elements of M^n and S^{m-n-1} at a point x respectively.

The purpose of this paper is to establish the following theorem.

THEOREM.

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(1.3) $$K := \left\{ \int_{S^{m-n-1}} \left[H_1(e)^2 - H_2(e) \right]^{n/2} d\sigma_{m-n-1} \right\} dV_n$$

is a local conformal invariant of the submanifold M^n immersed in E^m, and

(1.4) $$\int_{M^n} K$$

is a global conformal invariant of a compact oriented M^n in E^m.

This Theorem is due to W. Blaschke [1] for $m = 3$, $n = 2$, and due to B. Y. Chen [2] for $n = 2$ and a general m. Moreover, for a compact oriented M^2, by using the well-known Gauss-Bonnet formula from (1.4), it follows that $\int_{M^2} H_1^2\, dV_2$ is a global conformal invariant (J. H. White [4], B. Y. Chen [2]).

2. **Proof of the Theorem.** It is obvious that K is invariant under similarity transformations, so that it suffices to show that K is invariant under an inversion π on E^m, whose center does not lie on the submanifold M^n.

Choose the center of the inversion π to be the origin of a coordinate system in the Euclidean space E^m, and let x, $\bar{x}$ be the position vectors of a pair of corresponding points of the submanifold M^n and its image submanifold $\overline{M}^n$ under π. Then the definition of an inversion implies

(2.1) $$\bar{x} = (c^2 r^{-2}) x, \quad r^2 = x \cdot x,$$

where c is the radius of the inversion π. By (2.1) we readily obtain

(2.2) $$d\bar{x} = (c^2 r^{-2}) dx - 2(c^2 r^{-3} dr) x,$$

(2.3) $$d\bar{x} \cdot d\bar{x} = (c^4 r^{-4}) dx \cdot dx.$$

Let $e_{n+1}, \ldots, e_m$ be any $m - n$ mutually orthogonal unit normal vectors of M^n at x. Then from (2.2) it is easy to see that

(2.4) $$\bar{e}_\alpha = 2r^{-2}(x \cdot e_\alpha) x - e_\alpha, \qquad \alpha = n+1, \ldots, m,$$

are $m - n$ mutually orthogonal unit normal vectors of $\overline{M}^n$ at $\bar{x}$. Similarly, if e is a general unit normal vector of M^n at x, then

(2.5) $$\bar{e} = 2r^{-2}(x \cdot e) x - e$$

is a unit normal vector of $\overline{M}^n$ at $\bar{x}$. Since e can be written as

(2.6) $$e = \sum_{\alpha=n+1}^{m} a_\alpha e_\alpha, \quad \sum_{\alpha=n+1}^{m} a_\alpha^2 = 1,$$

we have

(2.7) $$\bar{e} = \sum_{\alpha=n+1}^{m} a_\alpha \bar{e}_\alpha.$$

Thus, if the vector e moves over the sphere S^{m-n-1} of M^n at x, then the vector $\bar{e}$ moves over the $(m - n - 1)$-dimensional sphere $\bar{S}^{m-n-1}$ of unit normal vectors of $\overline{M}^n$ at $\bar{x}$.

By means of (2.2) and (2.5) we obtain

$$d\bar{x}\cdot d\bar{e} = 2c^2r^{-4}(x\cdot e)dx\cdot dx - (c^2r^{-2})dx\cdot de, \tag{2.8}$$

and therefore, in consequence of (2.3),

$$d\bar{x}\cdot d\bar{e} + \lambda d\bar{x}\cdot d\bar{x} = -\frac{c^2}{r^2}\left[dx\cdot de + \left(-\frac{2(x\cdot e)}{r^2} - \lambda\frac{c^2}{r^2}\right)dx\cdot dx\right]. \tag{2.9}$$

Let $d\bar{V}_n$ be the volume element of the submanifold $\bar{M}^n$ at a point $\bar{x}$, and $\bar{h}_1(\bar{e}), \ldots, \bar{h}_n(\bar{e})$ be the principal curvatures of $\bar{M}^n$ at $\bar{x}$ with respect to $\bar{e}$. From (2.3), (2.9) and (1.1) it follows that

$$d\bar{V}_n = (c/r)^{2n}dV_n, \tag{2.10}$$

$$\bar{h}_i(e) = -c^{-2}r^2h_i(e) - 2c^{-2}(x\cdot e), \qquad i = 1, \ldots, n. \tag{2.11}$$

Thus by (1.2) and its corresponding equation for $\bar{M}^n$ we have

$$\bar{H}_1(\bar{e}) = -c^{-2}r^2H_1(e) - 2c^{-2}(x\cdot e), \tag{2.12}$$

$$\bar{H}_2(\bar{e}) = c^{-4}r^4H_2(e) + 4c^{-4}r^2(x\cdot e)H_1(e) + 4c^{-4}(x\cdot e)^2, \tag{2.13}$$

which, together with (2.10), immediately imply

$$\left[\bar{H}_1(\bar{e})^2 - \bar{H}_2(\bar{e})\right]^{n/2}d\bar{V}_n = \left[H_1(e)^2 - H_2(e)\right]^{n/2}dV_n. \tag{2.14}$$

Integrating both sides of (2.14) and using (1.3) we obtain

$$\bar{K} = K, \tag{2.15}$$

where $\bar{K}$ is defined by

$$\bar{K} = \left\{\int_{\bar{S}^{m-n-1}}\left[\bar{H}_1(e)^2 - \bar{H}_2(e)\right]^{n/2} d\bar{\sigma}_{m-n-1}\right\} dV_n, \tag{2.16}$$

$d\sigma_{m-n-1}$ being the volume element of S^{m-n-1} at x. If M^n is compact and oriented, then by integrating both sides of (2.15) over M^n we have

$$\int_{M^n}\bar{K} = \int_{M^n}K. \tag{2.17}$$

Hence our Theorem is proved.

It should be noted that a hypersphere of E^m has vanishing invariant K since the principal curvature $h_1(e), \ldots, h_n(e)$ of the hypersphere with respect to the unique unit normal vector e at every point x is equal.

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