CONFORMAL INVARIANTS OF SUBMANIFOLDS
CHUAN-CHIH HSIUNG and LARRY R. MUGRIDGE

Abstract. A local conformal invariant and a global conformal invariant of a submanifold immersed in a Euclidean space are derived.

1. Introduction. It is well known (Haantjes [3]) that every conformal mapping \( f \) on a Euclidean \( m \)-space \( E^m \) can be decomposed into a product of similarity transformations (i.e., Euclidean motions and homotheties) and inversions \( \{ \pi_i \} \). Let \( x: M^n \to E^m \) be an \( n \)-dimensional submanifold immersed in \( E^m \). For simplicity we shall write \( x(M^n) \) as \( M^n \). A quantity on \( M^n \) is a conformal invariant if it is invariant under the conformal mappings of \( E^m \), for which the center of every inversion does not lie on \( M^n \).

Let \( e \) be a unit normal vector of \( M^n \) at a point \( x \). Then the first fundamental form of \( M^n \) at \( x \) and the second fundamental form of \( M^n \) at \( x \) with respect to \( e \) are respectively defined to be

\[
I = dx \cdot dx, \quad II(e) = -dx \cdot de,
\]

where \( dx \) and \( de \) are vector-valued linear forms on \( M^n \), and the dot denotes the inner product of two vectors in \( E^m \); actually the form \( I \) is the Riemannian metric on \( M^n \) induced by the immersion. The eigenvalues \( h_1(e), \ldots, h_n(e) \) of \( II(e) \) relative to \( I \) are called the principal curvatures of \( M^n \) at the point \( x \) with respect to \( e \), and the \( r \)th mean curvature of \( M^n \) at \( x \) with respect to \( e \) is defined to be the \( r \)th elementary symmetric function of \( h_1(e), \ldots, h_n(e) \) divided by the number of terms, i.e.,

\[
H_r(e) = \frac{1}{\binom{n}{r}} \sum_{i_1, \ldots, i_r=1}^{n} h_{i_1}(e) \cdots h_{i_r}(e), \quad r = 1, \ldots, n,
\]

where \( \binom{n}{r} \) is the binomial coefficient.

Let \( B \) be the bundle of unit normal vectors of \( M^n \), so that a point of \( B \) is a pair \( (x, e) \). Then \( B \) is a bundle of \( (m-n-1) \)-dimensional spheres \( S^{m-n-1} \) of unit normal vectors over \( M^n \) and is a manifold of dimension \( m - 1 \). Let \( dV_n \) and \( d\sigma_{m-n-1} \) be the volume elements of \( M^n \) and \( S^{m-n-1} \) at a point \( x \) respectively.

The purpose of this paper is to establish the following theorem.

**Theorem.**
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(1.3) \[ K := \left( \int_{S^{m-n-1}} \left[ H_1(e)^2 - H_2(e) \right]^{n/2} \, d\sigma_{m-n-1} \right) dV_n \]
is a local conformal invariant of the submanifold \( M^n \) immersed in \( E^m \), and

(1.4) \[ \int_{M^n} K \]
is a global conformal invariant of a compact oriented \( M^n \) in \( E^m \).

This Theorem is due to W. Blaschke [1] for \( m = 3, n = 2 \), and due to B. Y. Chen [2] for \( n = 2 \) and a general \( m \). Moreover, for a compact oriented \( M^2 \), by using the well-known Gauss-Bonnet formula from (1.4), it follows that \( \int_{M^2} H_1^2 \, dV_2 \) is a global conformal invariant (J. H. White [4], B. Y. Chen [2]).

2. Proof of the Theorem. It is obvious that \( K \) is invariant under similarity transformations, so that it suffices to show that \( K \) is invariant under an inversion \( \pi \) on \( E^m \), whose center does not lie on the submanifold \( M^n \).

Choose the center of the inversion \( \pi \) to be the origin of a coordinate system in the Euclidean space \( E^m \), and let \( x, \tilde{x} \) be the position vectors of a pair of corresponding points of the submanifold \( M^n \) and its image submanifold \( \tilde{M}^n \) under \( \pi \). Then the definition of an inversion implies

(2.1) \[ \tilde{x} = (c^2r^{-2})x, \quad r^2 = x \cdot x, \]
where \( c \) is the radius of the inversion \( \pi \). By (2.1) we readily obtain

(2.2) \[ d\tilde{x} = (c^2r^{-2})dx - 2(c^2r^{-3}dr)x, \]

(2.3) \[ d\tilde{x} \cdot dx = (c^4r^{-4})dx \cdot dx. \]

Let \( e_{n+1}, \ldots, e_m \) be any \( m - n \) mutually orthogonal unit normal vectors of \( M^n \) at \( x \). Then from (2.2) it is easy to see that

(2.4) \[ \tilde{e}_\alpha = 2r^{-2}(x \cdot e_\alpha)x - e_\alpha, \quad \alpha = n + 1, \ldots, m, \]
are \( m - n \) mutually orthogonal unit normal vectors of \( \tilde{M}^n \) at \( \tilde{x} \). Similarly, if \( e \) is a general unit normal vector of \( M^n \) at \( x \), then

(2.5) \[ \tilde{e} = 2r^{-2}(x \cdot e)x - e \]
is a unit normal vector of \( \tilde{M}^n \) at \( \tilde{x} \). Since \( e \) can be written as

(2.6) \[ e = \sum_{\alpha=n+1}^{m} a_\alpha e_\alpha, \quad \sum_{\alpha=n+1}^{m} a_\alpha^2 = 1, \]
we have

(2.7) \[ \tilde{e} = \sum_{\alpha=n+1}^{m} a_\alpha \tilde{e}_\alpha. \]

Thus, if the vector \( e \) moves over the sphere \( S^{m-n-1} \) of \( M^n \) at \( x \), then the vector \( \tilde{e} \) moves over the \((m - n - 1)\)-dimensional sphere \( \tilde{S}^{m-n-1} \) of unit normal vectors of \( \tilde{M}^n \) at \( \tilde{x} \).

By means of (2.2) and (2.5) we obtain
(2.8) \[ d\mathbf{x} \cdot d\mathbf{e} = 2c^2r^{-4}(x \cdot e)dx \cdot dx - (c^2r^{-2})dx \cdot de, \]
and therefore, in consequence of (2.3),

(2.9) \[ d\mathbf{x} \cdot d\mathbf{e} + \lambda d\mathbf{x} \cdot d\mathbf{x} = -\frac{c^2}{r^2} \left[ dx \cdot de + \left( -\frac{2(x \cdot e)}{r^2} - \lambda \frac{c^2}{r^2} \right) dx \cdot dx \right]. \]

Let \( d\vec{V}_n \) be the volume element of the submanifold \( \overline{M}^n \) at a point \( \vec{x} \), and \( h_1(\vec{e}), \ldots, h_n(\vec{e}) \) be the principal curvatures of \( \overline{M}^n \) at \( \vec{x} \) with respect to \( \vec{e} \). From (2.3), (2.9) and (1.1) it follows that

(2.10) \[ d\vec{V}_n = (c/r)^2 n dV_n, \]

(2.11) \[ h_i(\vec{e}) = -c^{-2}r^2h_i(e) - 2c^{-2}(x \cdot e), \quad i = 1, \ldots, n. \]

Thus by (1.2) and its corresponding equation for \( \overline{M}^n \) we have

(2.12) \[ \overline{H}_1(\vec{e}) = -c^{-2}r^2H_1(e) - 2c^{-2}(x \cdot e), \]

(2.13) \[ \overline{H}_2(\vec{e}) = c^{-4}r^4H_2(e) + 4c^{-4}r^2(x \cdot e)H_1(e) + 4c^{-4}(x \cdot e)^2, \]

which, together with (2.10), immediately imply

(2.14) \[ \left[ \overline{H}_1(\vec{e})^2 - \overline{H}_2(\vec{e}) \right]^{n/2} d\vec{V}_n = \left[ H_1(e)^2 - H_2(e) \right]^{n/2} dV_n. \]

Integrating both sides of (2.14) and using (1.3) we obtain

(2.15) \[ \overline{K} = K, \]

where \( \overline{K} \) is defined by

(2.16) \[ \overline{K} = \left\{ \int_{S^{m-n-1}} \left[ H_1(e)^2 - H_2(e) \right]^{n/2} d\sigma_{m-n-1} \right\} dV_n, \]

\( d\sigma_{m-n-1} \) being the volume element of \( S^{m-n-1} \) at \( x \). If \( M^n \) is compact and oriented, then by integrating both sides of (2.15) over \( M^n \) we have

(2.17) \[ \int_{M^n} \overline{K} = \int_{M^n} K. \]

Hence our Theorem is proved.

It should be noted that a hypersphere of \( E^m \) has vanishing invariant \( K \) since the principal curvature \( h_1(e), \ldots, h_n(e) \) of the hypersphere with respect to the unique unit normal vector \( e \) at every point \( x \) is equal.

REFERENCES


DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY, BETHLEHEM, PENNSYLVANIA 18015
DEPARTMENT OF MATHEMATICS, KUTZTOWN STATE COLLEGE, KUTZTOWN, PENNSYLVANIA 19530

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