PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 62, Number 2, February 1977

TWO NOTES ON NILPOTENCY AND STANDARD ALGEBRAS

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ABSTRACT. Two results on nilpotency which are known to hold for Jordan and alternative algebras are shown to hold for standard algebras as well. These results in the known cases arose when developing a Cartan theory for these classes.

1. **Introduction.** If N is a nil subalgebra of a J-algebra A and if $[A,A]$ is semicompletely alternative, then a result of Block [2] shows that the enveloping associative algebra $U(A,N)^*$ of the set $U(A,N)$ of right and left multiplications of A by elements of N is nilpotent. For standard algebras this result will be sharpened to a result which is known to hold for Jordan and alternative algebras. This is Theorem 1. As an application, a Cartan theory for standard algebras, of characteristic not 3, paralleling the known ones for Jordan and alternative algebras can be developed using the same operator introduced by Foster [3] and a refinement of the Pierce decomposition [8] using identities in [9, p. 205]. This includes a theory of a-nilpotence, the existence of Cartan subalgebras except over fields with relatively few elements, a relation with the generic trace and the usual conjugacy theorem using the derivation $D(x,y) = [L_x + R_x, L_y + R_y]$. The standard algebra development follows the known ones closely, hence is not included. An exception is the standard algebra version of a lemma of Schafer which leads to Engel's theorem. This is Theorem 2.

2. **The results.** A finite dimensional algebra A over a field Φ of characteristic not 2 is called standard if the following identities hold in A

$$(wx, y, z) + (xz,y,w) + (wz,y,x) = 0, \tag{1}$$

$$(x,y,z) + (z,x,y) - (x,z,y) = 0, \tag{2}$$

$$(x,y,x^2) = 0 \tag{3}$$

where (r, s, t) is the associator $(r,s,t) = (rs)t - r(st)$. Note that (3) is a consequence of (1) if characteristic $\Phi \neq 3$. Standard algebras are discussed in [1], [7] and [9]. In particular, standard algebras are flexible, Jordan admissible

Received by the editors January 10, 1975 and, in revised form, March 24, 1975.
AMS (MOS) subject classifications (1970). Primary 17A15, 17A30.

and Lie admissible and the class includes all commutative Jordan algebras as well as associative algebras.

Let A be a standard algebra and let A^+ be the associated Jordan algebra whose space is the same as that of A and multiplication is defined by $a \cdot b = (ab + ba)/2$. Let B be a subalgebra of A, $U(A,B)$ be the collection of all right and left multiplications of A by elements of B and let $U(A,B)^*$ be the enveloping associative algebra of $U(A,B)$. For $x \in B$, let R_x^+ be defined by $yR_x^+ = y \cdot x$ for all $y \in A$ and let d_x be defined by $yd_x = [y,x] = yx - xy$ for all $y \in A$. The identity

$$cd = \tfrac{1}{2}[c,d] + c \cdot d \tag{4}$$

in A yields that a subspace of A which is invariant under R_x^+ and d_x is also invariant under R_x and L_x. Furthermore Schafer has noted [9, p. 203] that

$$[xy,z] = [x,z]y + x[y,z] \tag{5}$$

is an identity in any standard algebra. It follows that d_x is a derivation of A and hence of A^+.

THEOREM 1. *Let A be a standard algebra and let B be a subalgebra of A and let N be the nilpotent radical of B. Then $U(A,N)$ is contained in the radical of $U(A,B)^*$.*

PROOF. Let R^+ be the subalgebra of $U(A^+,B^+)^*$ generated by all elements of the form $R_{x_1}^+ \cdots R_{x_n}^+$ where at least one of the $x_i \in N$. Then R^+ is nilpotent by [5, Theorem 1]. Let $M_i = A(R^+)^i$, $i = 0, \ldots$. Then for some m, $M_m = 0$. Note that $(R^+)^i$ is generated by all elements of the form $R_{x_1}^+ \cdots R_{x_n}^+$ where at least i of the $x_1, \ldots, x_n$ are in N. Clearly $M_i R_x^+ \subseteq M_i$ for each $x \in B$. Also $M_i d_x \subseteq M_i$ since d_x is a derivation of A^+. Hence M_i is invariant under $U(A,B)$, and $U(A,B)^*$ acts on $\overline{M}_i = M_{i-1}/M_i$. Let R be the ideal of $U(A,B)^*$ generated by $U(A,N)$. Then R is generated by $\pi = \{$products of left and right multiplications of A by elements of B such that at least one of the multiplications is by an element of $N\}$. Let R_i and π_i be, respectively, the sets of linear transformations induced by R and π on $\overline{M}_i$. To show that R is nilpotent it suffices to prove that each R_i is nilpotent. We claim that $M_{i-1}R \subseteq [N, M_{i-1}] + M_i$. By using (4) it follows that $M_{i-1}\,U(A,N) \subseteq [N, M_{i-1}] + M_i$. Using (4) again,

$$\begin{aligned} B([N,M_{i-1}] + M_i) &\subseteq [B,[N,M_{i-1}] + M_i] + B \cdot ([N,M_{i-1}] + M_i) \\ &\subseteq [B,[N,M_{i-1}]] + [B,M_i] + B \cdot [N,M_{i-1}] + B \cdot M_i. \end{aligned}$$

The second and fourth terms on the right-hand side are contained in M_i since M_i is B-invariant. Since A is Lie admissible, the Jacobi identity yields that $[B,[N,M_{i-1}]] \subseteq [N,M_{i-1}]$. Now consider $B \cdot [N,M_{i-1}]$ and let $c \in B$, $x \in N$ and $w \in M_{i-1}$. Then, using (5),

$$
\begin{aligned}
c \cdot [x,w] &= \tfrac{1}{2}(c[x,w] + [x,w]c) \\
&= \tfrac{1}{2}([cx,w] - [c,w]x + [xc,w] - x[c,w]) \\
&= \tfrac{1}{2}([cx,w] + [xc,w]) - x \cdot [c,w] \in [N, M_{i-1}] + M_i .
\end{aligned}
$$

Hence $D_i \equiv [N, M_{i-1}] + M_i$ is invariant under L_x, $x \in B$, and similarly it is invariant under R_x, $x \in B$. Hence $M_{i-1}R \subseteq D_i$ and D_i is B-invariant by the foregoing. Now to show that R_i acts nilpotently on $\overline{M}_i$, it suffices to verify that R_i acts nilpotently on $\overline{D}_i = D_i/M_i$. Since commutators of A are contained in the nucleus [7, p. 335, (3)],

$$\overline{D}_i(\overline{L}_x\overline{L}_y - \overline{L}_{yx}) = 0, \quad \overline{D}_i(\overline{R}_x\overline{R}_y - \overline{R}_{xy}) = 0 \quad \text{and} \quad \overline{D}_i(\overline{R}_x\overline{L}_y - \overline{L}_y\overline{R}_x) = 0$$

for all $x, y \in B$. Hence each element of $\{\pi_i | \overline{D}_i\}$ can be written as one of the following: $\overline{L}_x$, $\overline{R}_x$, $\overline{L}_x\overline{R}_y$ or $\overline{L}_y\overline{R}_x$ where $x \in N$, $y \in B$, all of which are nilpotent. Furthermore $\{\pi_i | \overline{D}_i\}$ is a weakly closed set of linear transformations using the multiplicative semigroup $U(A,B)^*$. Hence $\{\pi_i | \overline{D}_i\}^* = R_i | \overline{D}_i$ is nilpotent by Jacobson's refinement to Engel's theorem [4, p. 33] and the proof is complete.

We use the same definitions of a-nilpotency as introduced by Foster [3]. Hence for $b_1, b_2, b_3 \in A$, let

$$a(b_1,b_2,b_3) = b_3(b_1b_2) + (b_2b_1)b_3 - (b_2b_3)b_1 - b_1(b_3b_2). \tag{6}$$

Since A is standard

$$a(b_1,b_2,b_3) = [b_2,[b_1,b_3]] + 2(b_1,b_2,b_3). \tag{7}$$

Let $S(b_2,b_3)$ be the linear transformation defined on A by $xS(b_2,b_3) = a(x,b_2,b_3)$. If $b \in A$, then b is called a-nilpotent if $S(b,b)$ is a nilpotent linear transformation. The usual Engel theorem asserts that if each $w \in A$ is a-nilpotent, then A is a-nilpotent. The standard algebra version follows the usual procedure except for the following

THEOREM 2. *Let A be a standard algebra with unity* 1, *over a field of characteristic not* 3. *If $w \in A$ is a-nilpotent, then every $S(w^i, w^j)$, $i, j = 0, 1, \ldots,$ is nilpotent.*

PROOF. First note that $S(b,c)$ may be written, using (6) and (7), as

$$S(b,c) = R_bL_c + L_bR_c - L_{bc} - R_{cb} \tag{8}$$

or

$$S(b,c) = R_cL_b - L_cL_b - R_cR_b + L_cR_b + 2(R_bR_c - R_{bc}). \tag{9}$$

Since $w^0 = 1$, the result is clear if $i = 0$ or $j = 0$. Assume $i, j > 0$. Let A_w be the associative algebra of all linear transformations of A generated by R_w, L_w, R_{w^2} and the identity linear transformation. Then A_w contains all R_b and L_b

where $b = w^i$, $i = 0, 1, \ldots$, by [1, p. 556, Lemma 5] and A_w is commutative by [1, p. 577, Theorem 5]. Hence the result will follow if every $S(w^i, w^j)$, $i, j > 0$, has the form $T_{ij} S(w, w)$ where $T_{ij} \in A_w$. We show this by induction on $h = i + j$. Since $S(w^i, w^j) = S(w^j, w^i)$ by (8), it suffices to show that $S(w^i, w^{j+1}) = T_{i,j+1} S(w, w)$ assuming that $S(w^k, w^l) = T_{kl} S(w, w)$ for all k, $l > 0$ and $k + l \leqslant i + j$. In order to obtain this we show that

$$S(x,yz) = R_z S(x,y) + R_y S(x,z) + S(y,z)L_x - S(y,z)R_x$$

where x, y, z are nonnegative integral powers of w. Now using (8) and (9)

$$\begin{aligned} &S(x,yz) - R_z S(x,y) - R_y S(x,z) - S(y,z)L_x + S(y,z)R_x \\ &\quad = R_{yz} L_x - L_{yz} L_x - R_{yz} R_x + L_{yz} R_x + 2(R_x R_{yz} - R_{x(yz)}) \\ &\qquad - R_z(R_y L_x - L_y L_x - R_y R_x + L_y R_x + 2(R_x R_y - R_{xy})) \\ &\qquad - R_y(R_z L_x - L_z L_x - R_z R_x + L_z R_x + 2(R_x R_z - R_{xz})) \\ &\qquad -(R_y L_z + L_y R_z - L_{yz} - R_{zy})L_x + (R_y L_z + L_y R_z - L_{yz} - R_{zy})R_x \\ &\quad \equiv H. \end{aligned}$$

All terms with a factor of 2 add together to give 0 by [9, p. 205, (12)]. Of the ten terms with a factor of L_x on the right, six add together to give 0, using that A_w is commutative. The same is true with L_x replaced by R_x. Hence

$$\begin{aligned} H = {} & R_{yz} L_x - R_z R_y L_x - R_y R_z L_x + R_{zy} L_x - R_{yz} R_x \\ & + R_z R_y R_x + R_y R_z R_x - R_{zy} R_x. \end{aligned}$$

For any $v \in A$, $vH = [(v,y,z) + (v,z,y), x] = 0$ since each associator in A commutes with each element in A by [7, (7)]. Hence $H = 0$. Now

$$S(w^i, w^j w) = R_w S(w^i, w^j) + R_{w^j} S(w^i, w) + S(w^j, w) L_{w^i} - S(w^j, w) R_{w^i}.$$

This reduction and the commutativity of A_w give the result.

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