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DENSE SUBGROUPS OF LIE GROUPS

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ABSTRACT. Let G be a dense analytic subgroup with compact center of an analytic group L. Then there exist closed vector subgroups W and U of G and a (CA) closed normal analytic subgroup M of G, which contains the center of G, such that $G = MWU$, $MW \cap U = M \cap W = \{e\}$, and WU is a closed vector subgroup of G. Moreover, $L = MW\overline{U}$, where MW is a closed normal analytic subgroup of L and $\overline{U}$ is a toral group, such that $MW \cap \overline{U}$ is finite.

1. **Introduction.** By an analytic group and an analytic subgroup of a Lie group, we mean a connected Lie group and a connected Lie subgroup, respectively. If G and H are Lie groups and φ is a one-to-one (continuous) homomorphism from G into H, φ will be called an immersion. φ will be called closed or dense, as $\varphi(G)$ is closed or dense in H. G_0 and $Z(G)$ will denote the identity component group and center of G, respectively.

If G is an analytic group, $A(G)$ will denote the Lie group of all (bicontinuous) automorphisms of G, topologized with the generalized compact-open topology. G will be called (CA) if $I(G)$, the Lie group of all inner automorphisms of G, is closed in $A(G)$. It is well known that G is (CA) if and only if its universal covering group is (CA).

If G is a normal analytic subgroup of an analytic group H, then each element h of H induces an automorphism of G, namely, $g \mapsto hgh^{-1}$. We will denote this homomorphism from H into $A(G)$ by ρ_{GH}. $I_H(h)$ will denote the inner automorphism of H determined by $h \in H$. More generally, if A is a subset of H, $I_H(A)$ will denote the set of all inner automorphisms of H determined by elements of A. $I_H(H)$ will be written as $I(H)$, and the mapping $h \mapsto I_H(h)$ of H onto $I(H)$ will be denoted by I_H.

If N is an analytic group and H is an analytic subgroup of $A(N)$, then $N \circledS H$ will denote the semidirect product of N and H. On the other hand, if G is an analytic group containing a closed normal analytic subgroup N and a closed analytic subgroup H, such that $G = NH$, $N \cap H = \{e\}$, and such that the restriction of ρ_{NG} to H is one-to-one, we will frequently identify G with $N \circledS \rho_{NG}(H)$ and H with $\rho_{NG}(H)$, that is, we may write $G = N \circledS H$.

In Zerling [3] we proved the following theorem.

MAIN STRUCTURE THEOREM. *Let G be a non-(CA) analytic group. Then there exist a (CA) analytic group M, a toral group T in $A(M)$, and a dense vector*

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subgroup V of T, such that:

(i) $H = M \circledS T$ *is a* (CA) *analytic group.*

(ii) G *is isomorphic to the dense analytic subgroup* $M \circledS V$ *of* H.

(iii) $Z(G)$ *is contained in* M.

(iv) $Z_0(G) = Z_0(H)$, *and* $\pi(Z(H))$ *is finite, where* π *is the natural projection of* H *onto* T. *Moreover, if* $G/Z(G)$ *is homeomorphic to Euclidean space, then* $Z(G) = Z(H)$.

(v) *Each automorphism* σ *of* G *can be extended to an automorphism* $\varepsilon(\sigma)$ *of* H, *such that* $\varepsilon: A(G) \to A(H)$ *is a closed immersion.*

We will now use this theorem in §2 to obtain our main results.

2. **Main results.**

LEMMA. *Let us maintain the notation of the main structure theorem and let* f: $G \to L$ *be a dense immersion of* G *into an analytic group* L. *If* $Z(G)$ *is compact, then* $f(M)$ *is closed in* L.

PROOF. Since G is non-(CA) we can appeal to Goto [1]: Let N be a maximal analytic subgroup of $I(G)$, which contains the commutator subgroup of $I(G)$ and is closed in $A(G)$. Then there is a closed vector subgroup V' of $I(G)$, such that $I(G) = NV'$, $N \cap V' = \{e\}$, and $\overline{I(G)} = N \cdot \overline{V}'$, where $T' = \overline{V}'$ is a toral group. Moreover, $N \cap T'$ is finite, and the space of $\overline{I(G)}$ is diffeomorphic to the product space $N \times T'$.

In the proof of the main structure theorem in Zerling [3], H is constructed in such a way that $\rho_{GH}(M) = N$, $\rho_{GH}(V) = V'$, and $\rho_{GH}(T) = T'$. Also, ρ_{GH} is 1-1 on T.

Since $Z(G)$ is of finite index in $Z(H)$ from Zerling [4, Lemma 2.1], $Z(H)$ is also compact. Consider the normal analytic subgroup $\overline{f(M)} \cdot f(V)$ of L. Since the inner automorphic action of $f(V)$ on $\overline{f(M)}$ is effective, we have the Lie group $P = \overline{f(M)} \circledS V$. The image of each one-parameter subgroup of V under ρ_{GP} is not closed in $A(G)$. Therefore, since G is dense in P, we see from Lemma 3.1 of Zerling [4] that the closure of V in $A(\overline{f(M)})$ is a toral group, which we will denote by T_1.

Now let $Q = \overline{f(M)} \circledS T_1$. Then $\rho_{GQ}(T_1) = T'$, and since $\tau_1 \cdot (m, v) \cdot \tau_1^{-1} = (\tau_1(m), v)$ for all (m, v) in G, we see that ρ_{GQ} is 1-1 on T_1. Since $\rho_{GH}(T) = T'$, and ρ_{GH} is 1-1 on T, we have that $\rho_{GQ}^{-1} \circ \rho_{GH}$ is an isomorphism of T onto T_1. Hence, $H = M \circledS T$ is a dense (CA) analytic subgroup of Q. Since $Z(H)$ is compact, we may appeal to van Est [2, Theorem 2.2.1] to conclude that $H \cong Q$. Hence, $f(M) = \overline{f(M)}$ and our theorem is proved.

THEOREM. *Let* f: $G \to L$ *be a proper dense immersion of an analytic group* G *into an analytic group* L. *Suppose* $Z(G)$ *is compact. Then there exist closed vector subgroups* W *and* U *of* G *and a* (CA) *closed normal analytic subgroup* M *of* G, *which contains* $Z(G)$, *such that*:

(i) $G = MWU$, $MW \cap U = M \cap W = \{e\}$, *and* WU *is a closed vector subgroup of* G.

(ii) $L = f(MW) \cdot \overline{f(U)}$, *where* $f(MW)$ *is a closed normal analytic subgroup of* L *and* $\overline{f(U)}$ *is a toral group. Moreover,* $f(MW) \cap \overline{f(U)}$ *is finite, and* $f(W) \cap (f(M) \cdot \overline{f(U)}) = f(W) \cap Z(L) = \{e\}$.

(iii) *If* L *is* (CA) *and* $Z(L)$ *is compact, then* $W = \{e\}$.

PROOF. Since $Z(G)$ is compact, we can conclude from van Est [2] that G is non-(CA). We will now maintain the notation of the main structure theorem, as well as the notation in the proof of the above lemma.

Since $Z(G)$ is compact, we know from the above lemma that there exists a maximal analytic subgroup J of G which contains M, such that $f(J)$ is closed in L. Then from Goto [1] there exists a closed vector subgroup U of G such that $G = JU$, $J \cap U = \{e\}$. Moreover, $L = f(J) \cdot \overline{f(U)}$, where $\overline{f(U)}$ is a toral group and $f(J) \cap \overline{f(U)}$ is finite.

In the proof of Goto's theorem [1], applied to $I(G)$, T' is a closed central subgroup of an arbitrarily fixed maximal compact subgroup K of $\overline{I(G)}$. We will assume that K has been selected so that it contains $\rho_{GL}(\overline{f(U)})$.

Now let $\pi\colon G \to V$ be the natural projection and let $W = \pi(J)$. Then W is a closed vector subgroup of V and since J contains M we see that $J = M \cdot W$, $M \cap W = \{e\}$. Therefore,

$$L = f(J) \cdot \overline{f(U)} = f(M) \cdot f(W) \cdot \overline{f(U)} = f(M) \cdot \overline{f(U)} \cdot f(W),$$

where $f(M) \cdot \overline{f(U)}$ is a closed normal analytic subgroup of L. Since $\overline{f(U)} \cap f(J)$ is finite and contained in $f(G)$, it is contained in $f(M)$. Hence, if $(f(M) \cdot \overline{f(U)}) \cap f(W) \neq \{e\}$, then $f(w) = f(m) \cdot x$, $x \in \overline{f(U)}$, and so $x = f(m)^{-1} \cdot f(w)$. Hence, $x \in \overline{f(U)} \cap f(J)$, which is contained in $f(M)$. By the uniqueness of the decomposition in J, we have $w = e$. So $L = (f(M) \cdot \overline{f(U)}) \cdot f(W)$, $(f(M) \cdot \overline{f(U)}) \cap f(W) = \{e\}$. Moreover, $f(W) \cap Z(L) = \{e\}$, since $f(J) \cap Z(L)$ is contained in $f(M)$, and $W \cap M = \{e\}$. Hence, $L = (f(M) \cdot \overline{f(U)}) \circledS f(W)$.

We now show that WU is a closed vector subgroup of $G = MWU$. Let $\varphi\colon W \to A(MU)$ be given by $\varphi(w)(mu) = w(mu)w^{-1}$. Since $W \cap Z(G) = \{e\}$, φ is an immersion and so $G = MU \circledS W$. Since the image of each one-parameter subgroup of W under I_G is not closed in $A(G)$, we see from Zerling [4] that $\overline{\varphi(W)}$ is a toral group.

Let $u \in U$. Then $I_G(\varphi(w) \cdot u) = I_G(wuw^{-1}) = I_G(u)$, since $I_G(w) \in T'$ commutes with $I_G(u) \in \rho_{GL}(\overline{f(U)})$ because of our selection of K above. Hence $\sigma(u) \cdot u^{-1} \in Z(G)$ for all $\sigma \in \overline{\varphi(W)}$. Since $Z(G)$ is a closed central subgroup of MU and each element of $\overline{\varphi(W)}$ keeps $Z(G)$ elementwise fixed, we see from Zerling [3, Lemma 2.2] that $\sigma(u) = u$ for all $\sigma \in \overline{\varphi(W)}$. Therefore, UW is a closed vector subgroup of G.

Now suppose that L is (CA) and $Z(L)$ is compact. As we did above, we can show that the closure of $f(W)$ in $A(f(M) \cdot \overline{f(U)})$ is a toral group, call it T_2. Then L is properly dense in $(f(M) \cdot \overline{f(U)}) \circledS T_2$. This is a contradiction from van Est [2]. Hence $W = \{e\}$. This completes the proof of our theorem.

3. **An example.** The following example shows that in the above theorem W need not be $\{e\}$, even if L is (CA). Let $G = MV$ be any non-(CA) analytic group and suppose that the dimension of V is at least two. Such a group G can easily be obtained by a slight modification of the example in Zerling [3].

We continue with our previous notation and let $S = G \circledS T'$. Since V is dense in T we can select an element $v_0 \in V$ such that v_0 generates a dense subgroup of T. Then $v_0' = I_G(v_0)$ generates a dense subgroup of T'. Let D denote the subgroup of S generated by $(v_0, v_0'^{-1})$. Then D is a free discrete central subgroup of S and $L = S/D$ is a (CA) analytic group for which $g \mapsto (g, e)D$ is a dense immersion f of G into L; see Zerling [4, the proof of Theorem 2.2].

Next let V_λ be the one-dimensional vector subgroup of V containing v_0, and let V_μ be a vector subgroup of V such that $V = V_\lambda \cdot V_\mu$, $V_\lambda \cap V_\mu = \{e\}$. Let δ: $S \to L$ be the canonical projection. Then $G = MV_\mu V_\lambda$, $f(MV_\mu)$ is closed in L and $\overline{f(V_\lambda)} = f(V_\lambda) \cdot \delta(T')$ is a toral group. $L = f(MV_\mu) \cdot \overline{f(V_\lambda)}$, and $f(MV_\mu) \cap \overline{f(V_\lambda)}$ is trivial. Hence $V_\mu = W \neq \{e\}$.

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