

DERIVATIONS, HOMOMORPHISMS, AND OPERATOR IDEALS

T. B. HOOVER¹

ABSTRACT. Let \mathfrak{A} be a C^* -algebra of operators on a Hilbert space, and let C_p be the Schatten p -ideal. It is shown that every derivation from \mathfrak{A} to C_p is inner. A similar argument shows that two C^* -homomorphisms which agree modulo C_p are equivalent.

It is well known [7] that every derivation on a von Neumann algebra is inner. In addition, if \mathfrak{A} is a C^* -algebra of operators and D is a derivation on \mathfrak{A} , then D extends to the von Neumann algebra generated by \mathfrak{A} and so D is "almost inner." Here the term *derivation* refers to a linear transformation D from \mathfrak{A} to \mathfrak{A} satisfying $D(AB) = AD(B) + D(A)B$ for each A and B in \mathfrak{A} , and D is *inner* provided there is a T in \mathfrak{A} satisfying $D(A) = AT - TA = D_T(A)$. One consequence of these results says that if a $*$ -automorphism ϕ of a von Neumann algebra \mathfrak{A} has a derivation as a logarithm, then ϕ is inner in the sense that there is a unitary operator U in \mathfrak{A} satisfying $\phi(A) = U^*AU$ for each A in \mathfrak{A} .

The derivation equation makes sense for linear maps D from the C^* -algebra \mathfrak{A} to a two sided \mathfrak{A} -module \mathfrak{J} . Here again it can be asked if such derivations are inner; that is, are they induced by an element of \mathfrak{J} as above? In fancier language, the question asks if the cohomology group $H^1(\mathfrak{A}, \mathfrak{J})$ is trivial [2]. In this paper we show that D is inner provided \mathfrak{A} is a C^* -subalgebra of the algebra $L(H)$ of all operators on a separable Hilbert space H , and \mathfrak{J} is the Schatten p ideal C_p , $1 \leq p < \infty$.

In contrast with the situation for derivations from an algebra to itself, our theorem does not directly give information about C^* -homomorphisms, but our technique of proof applies equally well to the study of homomorphisms. We show that if ϕ and ψ are representations of a C^* -algebra and if $\phi(A) - \psi(A)$ is in C_p with $\|\phi(A) - \psi(A)\|_p \leq \alpha\|A\| < \|A\|$ for each nonzero A in the algebra, then there is a unitary operator U with $1 - U$ in C_p and $\psi(A) = U^*\phi(A)U$. The theorem remains true, except for some finite-dimensional summands, if the norm condition is omitted.

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In the last section we discuss what happens in case \mathfrak{J} is all of $L(H)$ or the ideal K of compact operators.

I. Any derivation on a C^* -algebra \mathfrak{A} can be extended to the C^* -algebra obtained by adjoining an identity to \mathfrak{A} by defining $D(1) = 0$. Similarly a C^* -homomorphism ϕ on \mathfrak{A} can be extended by defining $\phi(1) = 1$. Therefore we consider only C^* -algebras which contain the identity, and if the C^* -algebra is a subalgebra of $L(H)$, we assume that the identity is the identity operator on H . With this in mind, we remark that every C^* -algebra (with identity) is generated by its group \mathfrak{U} of unitary elements.

In this section we deal with the von Neumann Schatten p -classes C_p , $1 \leq p < \infty$. We remark here that if $1 \leq p < p'$, then $C_p \subset C_{p'}$ and $\|T\|_{p'} \leq \|T\|_p$ for each T in C_p . ($\|\cdot\|_p$ denotes the norm on C_p .) The reader is referred to [1] for a discussion of these ideals. The primary tool for our first theorem is the Ryll-Nardzewski fixed point theorem [6]. This theorem states that if Q is a nonempty weakly compact convex subset of a locally convex Hausdorff linear topological space, and if G is a semigroup of weakly continuous affine maps on Q which is noncontracting, then there is a common fixed point for the maps in G . Here noncontracting means that for a, b in Q , $a \neq b$, there is a continuous seminorm ρ such that $\inf\{\rho(T(a) - T(b)) : T \in G\} > 0$. Application of the Ryll-Nardzewski theorem to derivation problems is suggested in [3].

THEOREM 1. *If \mathfrak{A} is a C^* -subalgebra of $L(H)$ which contains the identity operator, and if D is a derivation from \mathfrak{A} to C_p , $1 \leq p < \infty$, then D is inner. That is, there is a T in C_p such that $D = D_T$ and $\|T\|_p$ is less than or equal $\|D\|_p$, the norm of D as a linear transformation from \mathfrak{A} to C_p .*

PROOF. The operator D is continuous as a map from \mathfrak{A} to $L(H)$ [3] and so it is closed as a map from \mathfrak{A} to C_p . The continuity of D follows from the closed graph theorem.

First consider the case $p > 1$, so that C_p is a reflexive Banach space with dual space C_q , $1/p + 1/q = 1$. Let \mathfrak{U} be the unitary group of \mathfrak{A} , $K = \{U^*D(U) : U \in \mathfrak{U}\}$, and Q the closed convex hull of K in C_p . The set Q is bounded by $\|D\|_p$ and so, by the reflexivity of C_p , Q is weakly compact. For each U in \mathfrak{U} , define an affine map T_U on Q by $T_U(C) = U^*CU + U^*D(U)$. Then

$$\begin{aligned} T_U(V^*D(V)) &= U^*V^*D(V)U + U^*D(U) \\ &= U^*V^*(D(V)U + VD(U)) = U^*V^*D(VU). \end{aligned}$$

So T_U maps K to K and therefore Q onto Q . Furthermore,

$$\begin{aligned} T_U T_V(C) &= U^*[V^*CV + V^*D(V)]U + U^*D(U) \\ &= U^*V^*CVU + U^*V^*D(VU) = T_{UV}(C), \end{aligned}$$

so that $\{T_U : U \in \mathfrak{U}\}$ is a group. Clearly, the maps T_U are weakly continuous and if a and b are in Q ,

$$\|T_U(a) - T_U(b)\|_p = \|U^*(a - b)U\|_p = \|a - b\|_p$$

so that the group is noncontracting. Hence, by the Ryll-Nardzewski fixed point theorem, there is a common fixed point T for the T_U . That is,

$$T = T_U(T) = U^*TU + U^*D(U)$$

or $D(U) = UT - TU$ for each U in \mathfrak{U} . But \mathfrak{U} generates \mathfrak{A} , so $D = D_T$, and since T is in Q , $\|T\|_p \leq \|D\|_p$.

In case $p = 1$, then since $C_1 \subset C_q$ for $q > 1$, there is a T_q in C_q such that $D(A) = AT_q - T_qA$ for each A in \mathfrak{A} . Furthermore, if $q' > q$, $\|T_q\|_{q'} \leq \|T_q\|_q \leq \|D\|_q \leq \|D\|_1$. For each n , there is a sequence $\{T_{q_{n,m}} : m = 1, 2, \dots\}$ with $q_{n,m} > q_{n,m+1}$, which converges to an operator S_n in the weak* topology of $C_{1+1/n}$. Furthermore, the sequence $\{T_{q_{n+1,m}}\}$ can be chosen to be a subsequence of $\{T_{q_{n,m}}\}$. Note that $\|S_n\|_{1+1/n} \leq \|D\|_1$. But all of the S_n are the same, for if F is any finite rank operator,

$$\text{tr}(S_n F) = \lim_{m \rightarrow \infty} \text{tr}(T_{q_{n,m}} F) = \lim_{m \rightarrow \infty} \text{tr}(T_{q_{n+1,m}} F) = \text{tr}(S_{n+1} F).$$

(Here "tr" stands for the trace on C_1 .) Call the common value T . Clearly $D = D_T$, $T \in \bigcap_{q>1} C_q$ and $\|T\|_q \leq \|D\|_1$ for $q > 1$. Writing $T = UP$ in its polar decomposition, then P is in C_q for each $q > 1$ and $\|P\|_q \leq \|D\|_1$. It follows that $P^q \in C_1$ and $\|P^q\|_1 = \|P\|_q^q \leq \|D\|_1^q$. From here, it is easy to verify that P^q converges to P in the weak* topology on C_1 as q decreases to 1. Consequently P , and therefore T , is in C_1 and $\|P\|_1 = \|T\|_1 \leq \|D\|_1$.

We remark that our proof for the case $p > 1$ applies to any continuous derivation of \mathfrak{A} into a Banach \mathfrak{A} -module which is a reflexive Banach space. Proposition 3.7 of [2] also gives this result.

COROLLARY 2. *If \mathfrak{A} is a C^* -subalgebra of $L(H)$ and if D is a derivation from \mathfrak{A} to C_1 , then $\text{tr}(D(A)) = 0$ for each A in \mathfrak{A} .*

COROLLARY 3. *If \mathfrak{A} is a C^* -subalgebra of $L(H)$, and if B is an operator which commutes with \mathfrak{A} modulo C_p , $1 \leq p < \infty$, that is, if $AB - BA$ is in C_p for each A in \mathfrak{A} , then $B = A' + C$ where A' commutes with \mathfrak{A} and C is in C_p .*

PROOF. The operator B defines a derivation D_B from \mathfrak{A} to C_p . There is a C in C_p satisfying $D_B = D_C$, so $A' = B - C$ commutes with \mathfrak{A} .

COROLLARY 4. *An operator B commutes with a C^* -algebra \mathfrak{A} modulo C_p if and only if it commutes with the weak closure of \mathfrak{A} modulo C_p .*

II. Let \mathfrak{A} be any C^* -algebra and ϕ a representation of \mathfrak{A} on some Hilbert space H . If U is a unitary operator on H for which $1 - U$ is in C_p , then the representation ψ defined by $\psi(A) = U^* \phi(A)U$ for each A in \mathfrak{A} is such that $\psi - \phi$ is in C_p , that is, $\psi(A) - \phi(A)$ is in C_p for each A in \mathfrak{A} . Alternatively, let ϕ_1 and ϕ_2 be two n -dimensional representations; then $\phi \oplus \phi_1$ and $\phi \oplus \phi_2$ are

again representations which agree modulo C_p . We now show that these are the only two ways in which this can happen.

THEOREM 5. *Let \mathfrak{A} be a C^* -algebra with identity and suppose ϕ and ψ are representations of \mathfrak{A} on H such that $\phi - \psi$ is in C_p for some $p, 1 \leq p < \infty$. Then there is a partial isometry W for which $1 - W$ is in C_p and $\phi(A)W = W\psi(A)$ for each A in \mathfrak{A} . Furthermore, the initial space M of W reduces $\psi(\mathfrak{A})$ and the final space N reduces $\phi(\mathfrak{A})$, $\phi|_N$ is equivalent to $\psi|_M$, and M and N have the same finite codimension.*

PROOF. As with Theorem 1, we first assume that $p > 1$. Let $K = \{1 - \phi(U^*)\psi(U) : U \text{ unitary in } \mathfrak{A}\} \subset C_p$ and let Q be the closed convex hull of K . Since C_p is a reflexive Banach space, Q is compact in the weak topology on C_p . For C in Q and U in the unitary group \mathfrak{U} of \mathfrak{A} , define $T_U(C) = 1 - \phi(U^*)(1 - C)\psi(U)$. Then T_U is a weakly continuous affine map of Q to Q , $T_U T_V = T_{UV}$, and this action of the group \mathfrak{U} on Q is noncontracting. Therefore the Ryll-Nardzewski fixed point theorem applies, so there is a C in Q such that $T_U(C) = C$ for each U in \mathfrak{U} . That is,

$$C = 1 - \phi(U^*)(1 - C)\psi(U) \quad \text{or} \quad T = \phi(U^*)T\psi(U)$$

where $T = 1 - C$. But \mathfrak{U} generates \mathfrak{A} , so $\phi(A)T = T\psi(A)$ for each A in \mathfrak{A} .

Writing $T = WP$ according to its polar decomposition, we have $P^2 = T^*T = 1 - C^* - C + C^*C$ or

$$1 - P^2 = (1 - P)(1 + P) = C^* + C - C^*C \quad \text{is in } C_p$$

and

$$1 - P = (1 + P)^{-1}(C^* + C - C^*C) \quad \text{is in } C_p.$$

Therefore

$$1 - W = (1 - P) - (1 - T^*)W \quad \text{is in } C_p.$$

That W has the remaining desired properties follows by standard arguments.

The case $p = 1$ is proved by a weak* approximation argument using operators $C_p = 1 - T_p$ much as was done for Theorem 1.

In some cases, Theorem 5 gives unitary equivalence. Preserving the hypothesis and the notation of that theorem, we have:

COROLLARY 6. *If in addition \mathfrak{A} has no nonzero finite dimensional representations, then W is unitary.*

PROOF. The representations $\phi|_{N^\perp}$ and $\psi|_{M^\perp}$ are finite dimensional and so must be zero.

COROLLARY 7. *If $\|\phi(U) - \psi(U)\|_p \leq \alpha < 1$ for each U in \mathfrak{U} , then W is unitary.*

PROOF. From $\|\phi(U) - \psi(U)\|_p = \|1 - \phi(U^*)\psi(U)\|_p \leq \alpha$, it follows that $\|1 - T\|_p \leq \alpha$. Consequently $\|1 - T\| < 1$, T is invertible and W is unitary.

III. Derivations from a C^* -algebra into $L(H)$ and into the ideal K of compact operators have been studied elsewhere [2], [3], [4] and we have nothing new to add here. In this section we point out that many of the results about these derivations carry over to homomorphisms. Kadison and Ringrose [4] have shown that if \mathfrak{A} is the C^* -algebra generated by an amenable group of unitary operators, then every derivation from \mathfrak{A} into a dual Banach \mathfrak{A} -module is inner. In particular, this is true of derivations into $L(H)$. For any group G let $B(G)$ denote the Banach space of bounded functions on G with the supremum norm. A (left) invariant mean on $B(G)$ is a positive linear functional ϕ on $B(G)$ satisfying $\phi(1) = 1$ and $\phi({}_g f) = \phi(f)$ where for f in $B(G)$, g in G , ${}_g f$ is the function defined by ${}_g f(h) = f(gh)$. The group G is amenable if such a mean exists.

The following theorem generalizes a theorem of Lambert [5] since every abelian group is amenable.

THEOREM 9. *If G is an amenable group and if U_g and V_g are unitary representations of G on a Hilbert space H satisfying $\|U_g - V_g\| \leq \alpha < 1$ for each $g \in G$, then there is a unitary operator W on H such that $U_g = W^* V_g W$ for each g in G .*

PROOF. Let T be an operator satisfying $(Tx, y) = \phi(V_g^* U_g x, y)$ for each x and y in H , where ϕ is a left invariant mean on $B(G)$. Then

$$\begin{aligned} |((1 - T)x, y)| &= |\phi((1 - V_g^* U_g)x, y)| \leq \sup_{g \in G} \|1 - V_g^* U_g\| \|x\| \|y\| \\ &= \sup_{g \in G} \|V_g - U_g\| \|x\| \|y\| \leq \alpha \|x\| \|y\|. \end{aligned}$$

Therefore $\|1 - T\| < 1$ and T is invertible. Furthermore,

$$(V_g^* T U_g x, y) = \phi_h(V_g^* V_h^* U_h V_g x, y) = \phi_h(V_h^* U_h x, y) = (Tx, y)$$

so $V_g^* T U_g = T$ or $T U_g = V_g T$. Writing $T = WP$ according to its polar decomposition, W is unitary and satisfies the conclusions of the theorem.

THEOREM 10. *If a C^* -algebra \mathfrak{A} is generated by an amenable subgroup of its unitary group, and if ϕ and ψ are representations of \mathfrak{A} on H with $\|\phi(A) - \psi(A)\| \leq \alpha \|A\| < \|A\|$ for each nonzero A in \mathfrak{A} , then ϕ and ψ are equivalent.*

PROOF. If G is an amenable generating unitary group of \mathfrak{A} , then $U_g = \phi(g)$ and $V_g = \psi(g)$ are representations of G which are equivalent by Theorem 9. Consequently ϕ and ψ are equivalent.

It is certainly not the case that every derivation from a C^* -algebra into the ideal K of compact operators is inner. Johnson and Parrott [3], however, show that if \mathfrak{A} is a von Neumann algebra which does not contain a certain kind of type II_1 factor as a direct summand, then every derivation from \mathfrak{A} to K is

inner. Johnson and Parrott's arguments can be modified to study ultraweakly continuous representations of von Neumann algebras which are equal modulo the ideal K . These modifications parallel those made in the proof of Theorem 1 to get Theorem 5. For example, the following can be proved:

THEOREM 12. *If \mathfrak{A} is a von Neumann algebra with no type II_1 factor as a direct summand, and if ϕ and ψ are ultraweakly continuous representations of \mathfrak{A} such that $\phi(A) - \psi(A)$ is compact for each A in \mathfrak{A} , then there is a partial isometry W such that $1 - W$ is compact and $W\phi(A) = \psi(A)W$ for each A in \mathfrak{A} .*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII, HONOLULU, HAWAII 96822