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INDECOMPOSABLE COMPACT PERTURBATIONS OF THE BILATERAL SHIFT

DOMINGO A. HERRERO

ABSTRACT. Recent results of M. Radjabalipour and H. Radjavi assert that the sum of a normal operator N with spectrum on a smooth Jordan curve and a compact operator K in the Macaev ideal $\mathfrak{S}_\omega$ is decomposable provided the spectrum of $N + K$ does not fill the interior of the curve. Examples are given to show that this result cannot be essentially improved by taking K in a larger ideal.

1. Let $\mathcal{L}(\mathcal{H})$ be the algebra of all (bounded linear) operators on a complex Hilbert space $\mathcal{H}$. The *Macaev ideal* $\mathfrak{S}_\omega$ is the set of all compact operators K in $\mathcal{L}(\mathcal{H})$ such that $\sum_1^\infty \mu(n)/n < \infty$, where $\mu(1), \mu(2), \ldots,$ are the eigenvalues of $(K^*K)^{1/2}$ arranged in decreasing order and repeated according to multiplicity.

Compact perturbations of normal operators with spectrum on a smooth Jordan curve by an operator $K \in \mathfrak{S}_\omega$ are known to have a rich family of invariant subspaces. To make this precise, several definitions will be necessary:

An invariant subspace $\mathfrak{M}$ of $T \in \mathcal{L}(\mathcal{H})$ is a *maximal spectral subspace* of T if $\mathfrak{N} \subset \mathfrak{M}$ for all invariant subspaces $\mathfrak{N}$ of T such that the spectrum $\Lambda(T|\mathfrak{N})$ of the restriction of T to $\mathfrak{N}$ is contained in $\Lambda(T|\mathfrak{M})$. T is *decomposable* (in the sense of [1]) if for every finite covering G_j, $j = 1, 2, \ldots, n$, of $\Lambda(T)$ there exists a set of maximal spectral subspaces $\mathcal{H}_j, j = 1, 2, \ldots, n$, of T such that $\Lambda(T|\mathcal{H}_j) \subset G_j, j = 1, 2, \ldots, n$, and $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \cdots + \mathcal{H}_n$. Moreover, T is called *strongly decomposable* if $T|\mathfrak{M}$ is decomposable for every maximal spectral subspace $\mathfrak{M}$.

M. Radjabalipour and H. Radjavi [10]–[13] have improved a result of V. I. Macaev about compact perturbations of hermitian operators [9] by proving the following:

(i) Let T be the sum of an operator A having spectum on a C^2 Jordan curve J and a compact operator $K \in \mathfrak{S}_\omega$. Assume that $\|(z - A)^{-1}\| \leqslant C/d(z)$, where $d(z)$ is the distance from z to $\Lambda(A)$ and $C \geqslant 1$ is a constant

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independent of z, and that $\Lambda(T)$ does not fill the interior of J. Then T is strongly decomposable.

(ii) Assume that $\Lambda(T)$ is contained in a C^2 Jordan curve J and that there exist a positive number ε and a nonincreasing function M: $(0, \varepsilon) \to (0, \infty)$ such that $\int_0^\varepsilon \log^{(2)} M(t)\, dt < \infty$ ($\log^{(m)} x$ denotes the mth iterated logarithm). If $\|(z - T)^{-1}\| \leqslant M[d(z)]$ for $z \not\in J$, then T is strongly decomposable. This is true, in particular, if $M(t) = \exp(\exp t^{-p})$, $0 < p < 1$.

It will be shown that these two results are essentially sharp by proving that compact perturbations of the (unitary!) bilateral shift U in ℓ^2 (defined by $Ue_n = e_{n+1}$ for all n in the set $\mathbf{Z}$ of all integers, where $\{e_n\}$ is the canonical basis of ℓ^2) by an operator in a certain class of ideals which are only "slightly larger" than $\mathfrak{S}_\omega$ fail to be decomposable.

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2. Let $\{a_n\}$ be a bounded two-sided sequence of positive reals and define the bilateral weighted shift B in ℓ^2 by $Be_n = a_n e_{n+1}$, $n \in \mathbf{Z}$. If $\lim_{n\to\pm\infty} a_n = 1$, then $B - U$ is compact and $\Lambda(B)$ is the boundary ∂D of the unit disc $D = \{z: |z| < 1\}$.

Define $w_0 = 1$, $w_n = a_0, a_1 \cdots a_{n-1}$ and $w_{-n} = (a_{-1}a_{-2} \cdots a_{-n})^{-1}$ for all positive n. Then B is unitarily equivalent to multiplication by z on the space

$$L^2(\mathbf{w} = \{w_n\}) = \left\{ f(z) = \sum_{n\in\mathbf{Z}} b_n z^n \colon \|f\|^2 = \sum |b_n w_n|^2 < \infty \right\}$$

of formal Laurent series [3], [6]. Now we are in a position to construct the counterexamples.

THEOREM. *Let $\varphi = \{\varphi_n\}_1^\infty$ be a nonincreasing sequence of positive reals and assume that*: (i) *$\{\varphi_n \log en\}$ is nonincreasing*, (ii) *$n(\varphi_n - \varphi_{n+1}) \to 0$, and* (iii) *$\sum \varphi_n / n = \infty$. Let B be the bilateral weighted shift defined by $Be_n = a_n e_{n+1}$, where $a_n = w_{n+1}/w_n$, $w_0 = 1$ and $w_n = w_{-n} = \exp(n\varphi_n)$ for $n = 1, 2, \ldots$. Then*

(a) *Every nonzero invariant subspace $\mathfrak{M}$ of B is either invariant under B^{-1} and satisfies $\Lambda(B|\mathfrak{M}) = \partial D = \Lambda(B)$, or it is not invariant under B^{-1} and $\Lambda(B|\mathfrak{M}) = D^-$ is the closed unit disc.*

(b) *$K = B - U$ is a compact operator such that the eigenvalues of $(K^*K)^{1/2}$ (ordered as above indicated) satisfy $\mu_{2n-1}, \mu_{2n} \leqslant C\varphi_n$, where C is a constant independent of n.*

(c) *$\log^{(2)}\|(z - B)^{-1}\| \leqslant M/d(z)$, for a suitable constant M.*

PROOF. (a) This will follow by using the same kind of "sandwich theory" as in [4]:

Let $L^1(\mathbf{w}) = \{f(z) = \Sigma b_n z^n\colon \|f\|_1 + \Sigma|b_n w_n| < \infty\}$ and let $L^1(\sqrt{\mathbf{w}})$ be similarly defined, with w_n replaced by $\sqrt{w_n}$, $n \in \mathbf{Z}$ (the norm of an element of $L^1(\sqrt{\mathbf{w}})$ will be denoted by $\|\cdot\|''$). Then, for every finite sum $f(z) = \Sigma b_n w_n$, by the Cauchy-Schwarz inequality we have

$$\|f\|'' = \sum |b_n \sqrt{w_n}| = \sum |b_n w_n| / \sqrt{w_n}$$
$$\leqslant \Big(\sum |b_n w_n|^2\Big)^{1/2} \Big(\sum 1/w_n\Big)^{1/2} = C\|f\|,$$

where $C = (\Sigma\, 1/w_n)^{1/2} < \infty$ is a constant independent of f. It readily follows that $(1/C)\|f\|'' \leqslant \|f\| \leqslant \|f\|_1$ and, therefore, that $L^1(\mathbf{w}) \subset L^2(\mathbf{w}) \subset L^1(\sqrt{\mathbf{w}})$ and both inclusions are continuous.

It follows from (i) and (ii) that $\lim_{n\to\pm\infty} a_n = 1$ and, therefore, B is a compact perturbation of U and $\Lambda(B) = \partial D$ (see [3], [4], [6]). Moreover, it follows from (i)–(iii) and the results of [2], [4] that $L^1(\mathbf{w})$ and $L^1(\sqrt{\mathbf{w}})$ are actually algebras (under pointwise multiplication) of quasi-analytic functions defined on ∂D. I.e., $L^1(\sqrt{\mathbf{w}})$ is contained in $C^\infty(\partial D)$; if $f, g \in L^1(\mathbf{w})$ ($L^1(\sqrt{\mathbf{w}})$), then $f(z) \cdot g(z) \in L^1(\mathbf{w})$ ($L^1(\sqrt{\mathbf{w}})$, resp.) and the vanishing of $f(z) \in L^1(\sqrt{\mathbf{w}})$ together with all its derivatives at some point $\lambda \in \partial D$ implies that $b_n = 0$ for all n.

Let $\mathfrak{M}$ be a nonzero invariant subspace of B. If $\mathfrak{M}$ is not invariant under B^{-1}, then $0 \in \Lambda(B|\mathfrak{M})$ and it follows that $\Lambda(B|\mathfrak{M}) = D^-$; if $\mathfrak{M}$ is also invariant under B^{-1}, then $\Lambda(B|\mathfrak{M}) \subset \partial D$ (these two results easily follow from [7], [8]).

In the second case, the closure $\mathfrak{N}$ of $\mathfrak{M}$ in $L^1(\sqrt{\mathbf{w}})$ is an ideal of this algebra and we can proceed as in [4] in order to show that

$$\mathfrak{N} = \bigcap_{j=1}^{m} \left\{ f \in L^1(\sqrt{\mathbf{w}})\colon f(z_j) = f'(z_j) = \cdots = f^{(m_j - 1)}(z_j) = 0 \right\}$$

for a finite set of points $z_j \in \partial D$, $j = 1, 2, \ldots, m$, and positive integers m_j, $j = 1, 2, \ldots, m$, so that $\dim L^1(\sqrt{\mathbf{w}})/\mathfrak{N} = n = \Sigma_{j=1}^m m_j < \infty$. Clearly, $\mathfrak{M} \subset \mathfrak{N} \cap L^2(\mathbf{w})$ and for every $\lambda \in \partial D \setminus \{z_1, z_2, \ldots, z_m\}$ there exists a function $h_\lambda \in \mathfrak{M}$ such that $h_\lambda(\lambda) \neq 0$.

Let $\mathfrak{N}_0$ ($\mathfrak{N}''$) be the closure of $(B - \lambda)\mathfrak{M}$ in $L^2(\mathbf{w})$ (in $L^1(\sqrt{\mathbf{w}})$, resp.). Since $(B - \lambda)f(z) = (z - \lambda)f(z)$, it readily follows that $\mathfrak{N}'' = \mathfrak{N} \cap \{f \in L^1(\sqrt{\mathbf{w}})\colon f(\lambda) = 0\}$ has codimension 1 in $\mathfrak{N}$. In particular, $h_\lambda \in \mathfrak{M} \setminus \mathfrak{N}''$, a fortiori, $h_\lambda \notin \mathfrak{N}_0$ ($\mathfrak{N}_0 \subset \mathfrak{M} \cap \mathfrak{N}''$) and, therefore, $(B - \lambda)\mathfrak{M}$ is not dense in $\mathfrak{M}$. Hence, $\lambda \in \Lambda(B|\mathfrak{M})$.

Since this holds for all but finitely many λ's in ∂D, it is not difficult to conclude that $\partial D \subset \Lambda(B|\mathfrak{M})$ and, therefore, $\Lambda(B|\mathfrak{M}) = \partial D$, as promised.

If $\mathfrak{M}$ contains a nonzero function $L^1(\mathbf{w})$, it is easy to see that $\mathfrak{M} \cap L^1(\mathbf{w})$ is actually an ideal of finite codimension n in $L^1(\mathbf{w})$ and, a fortiori, that $\mathfrak{M}$ also has codimension n in $L^2(\mathbf{w})$. (The details are left to the reader. The author conjectured that $\mathfrak{M} \cap L^1(\mathbf{w}) \neq \{0\}$ for every invariant subspace $\mathfrak{M}$, but was unable to prove it.)

(b) This follows immediately from $Ke_n = (B - U)e_n = (a_n - 1)e_{n+1}$ (see [5]).

(c) Since U is normal, $\gamma(t) = \max\{\|(z - U)^{-1}\|: d(z) = t\} = 1/t$. On the other hand (b) and condition (i) show that $\mu_n \leqslant k/\log en$, for some constant $k > 0$ and for all $n = 1, 2, \ldots$, and, therefore, $\nu(t) = \max\{n: \mu_n > 1/t\} \leqslant e^{kt}$. Now we are in a position to apply Theorem 1 of [13], whence the result follows. □

3. The sequences

$$\left\{\varphi_n = \varphi_n(m) = \left[\log M_1 n \cdot \log^{(2)} M_2 n \cdot \cdots \cdot \log^{(m)} M_m n\right]^{-1}\right\},$$

where $M_1 = e$ and $M_{k+1} = \exp(M_k)$ for $k = 1, 2, \ldots, m - 1$ ($m = 1, 2, \ldots$), are concrete examples of sequences satisfying conditions (i)–(iii). Moreover, if $m_n \nearrow \infty$ and

$$\varphi_n = \psi_n(m_n) = \varphi_n(m_n)/\left[\log^{(m_n+1)} M_{m_n+1} n\right]^2$$

satisfies (iii), then it actually satisfies (i)–(iii).

Given an arbitrary increasing function $h(n)$ such that $h(1) = 1$ and $\lim_{n\to\infty} h(n) = \infty$, it is not difficult to find a sequence $\{m_n\}_{n=1}^{\infty}$ such that $\{\psi_n(m_n)\}$ satisfies (i)–(iii), but $\Sigma_1^\infty \psi_n(m_n)/[nh(n)] < \infty$. Now we can give a very precise meaning to our introductory sentence "these two results are essentially sharp":

Corollary. *Let $h(n)$ be an arbitrary increasing function such that $h(1) = 1$ and $\lim_{n\to\infty} h(n) = \infty$, and let $\mathfrak{S}_h = \{K \in \mathcal{L}(\ell^2)$: K is compact and $\Sigma\, \mu(n)/[nh(n)] < \infty\}$. Then there exists an operator K in the ideal $\mathfrak{S}_h$ such that $U + K$ is indecomposable, even when its resolvent satisfies the growth conditions of the Theorem.*

The proof follows immediately from the Theorem and the above observations.

4. **An example**. Let $\varphi_n = (\log en \cdot \log^{(2)} e^2 n)^{-1}$; then a straightforward computation shows that μ_n is of the same order of magnitude as φ_n ($n = 1, 2, \ldots$) and that, in this case, $\nu(t) \leqslant \exp(kt/\log t)$, whence (by applying once again Theorem 1 of [13]) we can improve the estimation (c) of the Theorem to

$$\log^{(2)}\|(z - B)^{-1}\| \leqslant M/\left[d(z) \cdot \log 1/d(z)\right],$$

for a suitable constant M and for $0 < d(z) < \frac{1}{2}$.

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Departmento de Matemáticas, Universidad Simón Bolívar, Caracas, Venezuela

Current address: Departamento de Matemáticas, Instituto Venezolano de Investigaciones Científicas, Caracas, Venezuela