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FIXED POINT THEOREMS FOR MAPPINGS WITH A CONTRACTIVE ITERATE AT A POINT[1]

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ABSTRACT. Let (X,d) be a complete metric space, $T: X \to X$, and $\alpha: [0,\infty)^5 \to [0,\infty)$ be nondecreasing with respect to each variable. Suppose that for the function $\gamma(t) = \alpha(t,t,t,2t,2t)$, the sequence of iterates γ^n tends to 0 in $[0,\infty)$ and $\lim_{t\to\infty}(t - \gamma(t)) = \infty$. Furthermore, suppose that for each $x \in X$ there exists a positive integer $n = n(x)$ such that for all $y \in X$,

$$d(T^n x, T^n y) \leqslant \alpha(d(x,T^n x), d(x,T^n y), d(x,y), d(T^n x,y), d(T^n y,y)).$$

Under these assumptions our main result states that T has a unique fixed point. This generalizes an earlier result of V. M. Sehgal and some recent results of L. Khazanchi and K. Iseki.

1. For a function $\gamma: [0,\infty) \to [0,\infty)$ denote by γ^n, $n = 0, 1, \ldots$, the nth iterate of γ. Before stating the main result we prove the following.

LEMMA . *Suppose that* $\gamma: [0,\infty) \to [0,\infty)$ *is nondecreasing. Then for every* $t > 0$, $\lim_{n\to\infty} \gamma^n(t) = 0$ *implies* $\gamma(t) < t$.

PROOF. Suppose that for some $t_0 > 0$ we have $\gamma(t_0) \geqslant t_0$. Hence, by the monotonicity of γ, $\gamma^n(t_0) \geqslant t_0$ for $n = 1, 2, \ldots$. This proves the lemma.

REMARK 1. Note that for every right continuous function $\gamma: [0,\infty) \to [0,\infty)$ such that $\gamma(t) < t$ for $t > 0$, $\lim_{n\to\infty} \gamma^n(t) = 0$.

THEOREM 1. *Let* (X,d) *be a complete metric space,* $T: X \to X$, $\alpha: [0,\infty)^5 \to [0,\infty)$, *and let* $\gamma(t) = \alpha(t,t,t,2t,2t)$ *for* $t \geqslant 0$.

Suppose that

1°. α *is nondecreasing with respect to each variable,*

2°. $\lim_{t\to\infty}(t - \gamma(t)) = \infty$,

3°. $\lim_{n\to\infty} \gamma^n(t) = 0$, $t > 0$,

4°. *for every* $x \in X$, *there exists a positive integer* $n = n(x)$ *such that for all* $y \in X$,

$$d(T^n x, T^n y) \leqslant \alpha(d(x,T^n x), d(x,T^n y), d(x,y), d(T^n x,y), d(T^n y,y)).$$

Then T *has a unique fixed point* $a \in X$ *and for each* $x \in X$, $\lim_{k\to\infty} T^k x = a$.

PROOF. First we shall show that for every $x \in X$, the orbit $\{T^i x\}_{i=0}^{\infty}$ is bounded.

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To prove this assertion we fix an $x \in X$, an integer s, $0 \leqslant s < n = n(x)$, and we put

$$u_k = d(x, T^{kn+s}x), \qquad k = 0, 1, \ldots,$$
$$h = \max(u_0, d(x, T^n x)).$$

By 2° there is a c, $c > h$, such that

$$t - \gamma(t) > h, \qquad t > c.$$

From the choice of c we have $u_0 < c$. Suppose that there exists a positive integer j such that $u_j \geqslant c$. Evidently, we may assume that $u_i < c$ for $i < j$. Hence, by the triangle inequality,

$$d(T^n x, T^{(j-1)n+s}x) \leqslant d(x, T^n x) + u_{j-1} < 2u_j,$$
$$d(T^{jn+s}x, T^{(j-1)n+s}x) \leqslant u_j + u_{j-1} < 2u_j.$$

Now, using 4° and 1°, we get

$$u_j = d(x, T^{jn+s}x) \leqslant d(T^n x, T^n T^{(j-1)n+s}x) + d(x, T^n x)$$
$$\leqslant \alpha(u_j, u_j, u_j, 2u_j, 2u_j) + h = \gamma(u_j) + h,$$

i.e. $u_j - \gamma(u_j) \leqslant h$ which together with $u_j > c$ contradicts to the choice of c. Therefore $u_j < c$ for $j = 0, 1, \ldots$, and, consequently, the orbit $\{T^i x\}_{i=0}^{\infty}$ is bounded.

Take an $x_0 \in X$ and put $n_0 = n(x_0)$. Define $\{x_k\}$ as follows

(1) $$x_{k+1} = T^{n_k}x_k, \qquad n_k = n(x_k), \ k = 0, 1, \ldots.$$

Evidently, $\{x_k\}$ is a subsequence of the orbit $\{T^i x_0\}_{i=0}^{\infty}$. We shall prove that $\{x_k\}$ is a Cauchy sequence.

Let k and i be positive integers. From (1) we have

$$x_{k+i} = T^{n_{k+i-1}+\cdots+n_k}x_k.$$

Denoting $s_0 = n_{k+i-1} + \cdots + n_k$, we can write

$$d(x_k, x_{k+i}) = d(x_k, T^{s_0}x_k).$$

For the simplicity of the notations we put $t_i = d(x_{k-1}, T^i x_{k-1})$. Denote by s_1 that of the numbers s_0, n_{k-1}, $s_0 + n_{k-1}$ for which t_{s_1} has the greatest value. Thus, by the triangle inequality, we have

$$d(T^{n_{k-1}}x_{k-1}, T^{s_0}x_{k-1}) \leqslant t_{n_{k-1}} + t_{s_0} \leqslant 2t_{s_1},$$
$$d(T^{n_{k-1}+s_0}x_{k-1}, T^{s_0}x_{k-1} \leqslant t_{n_{k-1}+s_0} + t_{s_0} \leqslant 2t_{s_1}.$$

Now 4° and 1° imply

$$d(x_k, T^{s_0}x_k) = d(T^{n_{k-1}}x_{k-1}, T^{n_{k-1}}T^{s_0}x_{k-1})$$
$$\leqslant \alpha(t_{s_1}, t_{s_1}, t_{s_1}, 2t_{s_1}, 2t_{s_1}) = \gamma(t_{s_1}),$$

i.e.,

$$d(x_k, T^{s_0}x_k) \leqslant \gamma(d(x_{k-1}, T^{s_1}x_{k-1})).$$

Repeating this procedure, we can find positive integers s_j, $j = 1, \ldots, k-1$, such that

$$d(x_{k-j}, T^{s_j}x_{k-j}) \leqslant \gamma(d(x_{k-j-1}, T^{s_{j+1}}x_{k-j-1})).$$

Hence, since γ is nondecreasing, we obtain

$$d(x_k, x_{k+i}) \leqslant \gamma^k(d(x_0, T^{s_k}x_0)) \leqslant \gamma^k(M),$$

where M denotes the diameter of the orbit $\{T^i x_0\}_{i=0}^{\infty}$. By 3°, $\lim_{k\to\infty}\gamma^k(M) = 0$. This proves that $\{x_k\}$ is a Cauchy sequence.

By the completeness of X there is an $a = \lim_{k\to\infty} x_k$, $a \in X$. We shall show that for $n = n(a)$, $T^n a = a$.

For an indirect proof suppose that $\varepsilon = d(T^n a, a) > 0$. Using the argument of the preceding part of the proof, we see that

$$\lim_{k\to\infty} d(T^n x_k, x_k) = 0.$$

Therefore, by the Lemma, there exists a k_0 such that

$$d(a, x_k) \leqslant \tfrac{1}{4}(\varepsilon - \gamma(\varepsilon)), \quad d(T^n x_k, x_k) \leqslant \tfrac{1}{4}(\varepsilon - \gamma(\varepsilon)), \qquad k \geqslant k_0.$$

Hence we get

$$\varepsilon = d(T^n a, a) \leqslant d(T^n a, T^n x_k) + d(T^n x_k, x_k) + d(x_k, a)$$
$$\leqslant \alpha(d(a, T^n a), d(a, T^n x_k), d(a, x_k), d(T^n a, x_k), d(T^n x_k, x_k))$$
$$+ \tfrac{1}{2}(\varepsilon - \gamma(\varepsilon)).$$

Since $d(a, T^n x_k) \leqslant d(a, x_k) + d(x_k, T^n x_k)$, $d(T^n a, x_k) \leqslant d(T^n a, a) + d(a, x_k)$, it follows that for $k \geqslant k_0$,

$$d(a, T^n x_k) \leqslant \tfrac{1}{2}(\varepsilon - \gamma(\varepsilon)) < \varepsilon, \qquad d(T^n a, x_k) \leqslant 2\varepsilon.$$

Therefore, by 1°,

$$\varepsilon \leqslant \alpha(\varepsilon, \varepsilon, \varepsilon, 2\varepsilon, 2\varepsilon) + \tfrac{1}{2}(\varepsilon - \gamma(\varepsilon)) = \tfrac{1}{2}(\varepsilon + \gamma(\varepsilon)) < \varepsilon,$$

which is a contradiction. Consequently, $T^n a = a$.

Suppose that there is a point $b \in X$, $b \neq a$, such that $T^n b = b$ with $n = n(a)$. Then by 4° and the Lemma

$$d(a, b) = d(T^n a, T^n b) \leqslant \alpha(0, d(a, b), d(a, b), d(a, b), 0)$$
$$\leqslant \gamma(d(a, b)) < d(a, b).$$

This contradiction proves that a is a unique fixed point of T^n.

Since $Ta = T^n Ta$, just proved uniqueness yields $Ta = a$. Now it is trivial that a is a unique fixed point of T.

To prove the last statement of Theorem 1 take an $x \in X$, an integer s, $0 \leqslant s < n = n(a)$, and put

$$a_k = d(a, T^{kn+s}x), \qquad k = 0, 1, \ldots.$$

Suppose that for some k, $a_k > a_{k-1}$. Then, using 4°, 1° and 3° (cf. the Lemma), we have

$$\begin{aligned} a_k &= d(T^n a, T^n T^{(k-1)n+s}x) \\ &\leqslant \alpha(0, a_k, a_{k-1}, a_{k-1}, d(T^{kn+s}x, T^{(k-1)n+s}x)) \\ &\leqslant \alpha(a_k, a_k, a_k, a_k, 2a_k) \leqslant \gamma(a_k) < a_k. \end{aligned}$$

This contradiction proves that $a_k \leqslant a_{k-1}$, $k = 1, 2, \ldots$. Hence, using 4° and 1°, we have

$$a_k = d(T^n a, T^{kn+s}x) \leqslant \alpha(a_{k-1}, a_{k-1}, a_{k-1}, a_{k-1}, 2a_{k-1}) \leqslant \gamma(a_{k-1})$$

for $k = 1, 2, \ldots$. This yields $a_k \leqslant \gamma^k(a_0)$ and, in view of 3°, $\lim_{k\to\infty} a_k = 0$. This completes the proof.

REMARK 2. Note that we have not assumed the continuity of T.

2. As a simple consequence of Theorem 1 we obtain the following

THEOREM 2. *Let (X,d) be a complete metric space, $T: X \to X$, and $\gamma: [0, \infty) \to [0, \infty)$. If γ is nondecreasing, $\lim_{t\to\infty}(t - \gamma(t)) = \infty$, $\lim_{k\to\infty}\gamma^k(t) = 0$ for $t > 0$, and for each $x \in X$ there is a positive integer $n = n(x)$ such that for all $y \in X$,*

$$d(T^n x, T^n y) \leqslant \gamma(d(x,y)),$$

then T has a unique fixed point $a \in X$. Moreover, for each $x \in X$, $\lim_{k\to\infty} T^k x = a$.

REMARK 3. Taking in Theorem 2 $\gamma(t) = ct$ with $0 < c < 1$, we obtain V. M. Sehgal's fixed point theorem in which the assumption of the continuity of T is removed (cf. [4]).

For $\alpha(t_1, \ldots, t_5) = at_1 + bt_2 + ct_3 + bt_4 + at_5$, Theorem 1 yields the following

THEOREM 3. *Let (X,d) be a complete metric space and let $T: X \to X$ satisfies the following condition: for each $x \in X$ there is a positive integer $n = n(x)$ such that for all $y \in X$,*

$$d(T^n x, T^n y) \leqslant a[d(x, T^n x) + d(y, T^n y)] + b[d(x, T^n y) + d(T^n x, y)] + cd(x,y)$$

where a, b, c are nonnegative and $3a + 3b + c < 1$ then T has a unique fixed point $p \in X$. Moreover, for every $x \in X$, $\lim_{k\to\infty} T^k x = p$.

Recently K. Iseki [2], generalizing the results of V. M. Sehgal [4] and L. Khazanchi [3], has obtained an analogous result but there T is assumed to be continuous and $4a + 4b + c < 1$.

REMARK 4. L. F. Guseman [1] noted that in [4] the continuity of T is superfluous. He also gave an interesting reformulation of Sehgal's result. In a similar way we can formulate our Theorems 1–3.

EXAMPLE. Let $X = [0, \infty)$, $d(x,y) = |x - y|$, $Tx = x/(1 + x)$, $\gamma(t) = t/(1 + t)$ for $x, y, t \in [0, \infty)$. We have

$$\lim_{n\to\infty} \gamma^n(t) = \lim_{n\to\infty} \frac{t}{1 + nt} = 0 \quad \text{for } t \geqslant 0, \ \lim_{t\to\infty} (t - \gamma(t)) = \infty$$

and

$$d(Tx, Ty) = \left|\frac{x}{x+1} - \frac{y}{y+1}\right| = \frac{|x - y|}{(1 + x)(1 + y)} \leqslant \frac{|x - y|}{1 + |x - y|} = \gamma(d(x,y)).$$

Thus all the assumptions of Theorem 2 are fulfilled, but that is not the case for Theorem 3. To see this suppose that there are nonnegative a, b, c satisfying conditions of Theorem 3. Then for $x = 0$ we obtain

$$\frac{y}{1 + ny} \leqslant a\left(y - \frac{y}{1 + ny}\right) + b\left(\frac{y}{1 + ny} + y\right) + cy, \qquad n = n(0), y > 0.$$

Hence $(a + b + c)/(1 - a + b) \geqslant 1/(1 + ny)$ for $y > 0$ and, consequently, $2b + c \geqslant 1$. This contradiction proves that Theorem 1 is stronger than the results of [1]—[4].

REMARK 5. Suppose that T: $X \to X$ and there is a point $p \in X$ such that $\{d(T^n x,p)\}$ tends to 0 uniformly in X. Fix an $a > 0$. Then for every $x \in X - \{p\}$ there exists a positive integer $n = n(x)$ such that for all $y \in X$, $d(T^n x, T^n y) \leqslant ad(x, T^n x)$. Using this remark one can easily construct an example of a mapping T satisfying all the conditions of Theorem 3 and such that for every n, T^n is discontinuous.

REFERENCES

1. L. F. Guseman, *Fixed point theorems for mappings with a contractive iterate at a point*, Proc. Amer. Math. Soc. **26** (1970), 615-618. MR **42** #919.

2. K. Iseki, *A generalization of Sehgal-Khazanchi's fixed point theorems*, Math. Seminar Notes Kobe Univ. **2** (1974), 1-9.

3. L. Khazanchi, *Results on fixed points in complete metric space*, Math. Japonicae (to appear).

4. V. M. Sehgal, *A fixed point theorem for mappings with a contractive iterate*, Proc. Amer. Math. Soc. **23** (1969), 631-634. MR **40** #3531.

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