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ON AHLFORS' "SECOND FUNDAMENTAL INEQUALITY"[1]

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ABSTRACT. The authors have earlier given a generalization of Ahlfors' "Second Fundamental Inequality" which reduced significantly the restrictions on the domains involved but retained the condition that certain determining functions for the domain be of bounded variation. In this paper it is shown that the condition of bounded variation can be replaced by that of finite 2/3-variation on any closed interval and an appropriate new formal expression is given.

1. The conformal mapping of strip domains was first treated in a unified manner by Ahlfors in his thesis [1]. He obtained estimates for certain associated geometrical quantities in terms of integrals which he called the First and Second Fundamental Inequalities. Recently the authors [4] showed that a very natural and effective approach to these results is obtained by using the method of the extremal metric, in particular estimating the modules of certain quadrangles associated with the strip domain from below and above in terms of the integral utilized by Ahlfors. In this way the proof of the First Fundamental Inequality (Ahlfors Distortion Theorem) becomes almost obvious. In his form of the Second Fundamental Inequality, Ahlfors imposed certain stringent requirements on the strip domain [1, p. 12]. We were able to weaken these assumptions substantially, in particular dropping conditions 1 and 2. However we still required that certain associated functions were of bounded variation. It is readily seen from examples that we cannot completely eschew some such condition but in this paper we will show that a substantially weaker one will suffice. We should point out that a version of the Second Fundamental Inequality, weakening some of the conditions, was obtained also by Ferrand and Dufresnoy [2], it being, however, less directly a generalization of the Ahlfors format.

2. DEFINITION. *Let $f(x)$ be a real-valued function defined on the interval $[x_1, x_2]$. By the α-variation ($\alpha > 0$) of $f(x)$ on $[x_1, x_2]$ we mean the least upper bound of the quantity*

$$\left\{ \sum_{j=1}^{n} |f(x^{(j)}) - f(x^{(j-1)})|^{\alpha^{-1}} \right\}^{\alpha}$$

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taken for all subdivisions $x_1 = x^{(0)} < x^{(1)} < \cdots < x^{(n-1)} < x^{(n)} = x_2$. *We denote this quantity by* $V^{(\alpha)}(f; x_1, x_2)$.

It should be observed that this concept has been studied to some extent from a technical point of view. Some references are found in [3].

THEOREM 1. *Let D be a simply-connected domain in the z-plane with boundary elements* P_1 *and* P_2 *such that for* $A < x < B$ *there exists a segment* $\sigma(x)$ *on* $\Re z = x$ *represented by* $-\theta_1(x) < y < \theta_2(x)$, $\theta_1(x), \theta_2(x) > 0$, *which is a cross cut of D separating* P_1 *and* P_2. *Let Q be a quadrangle whose domain is a component of* $D - (\sigma(x_1) \cup \sigma(x_2))$, $A < x_1 < x_2 < B$, *with* $\sigma(x_1), \sigma(x_2)$ *as a pair of opposite sides. Let the module of the quadrangle for the family of curves joining the complementary pair of sides be denoted by M. Let* $\theta(x) = \theta_1(x) + \theta_2(x)$. *Suppose* θ_1, θ_2 *have finite 2/3-variation on* $[x_1, x_2]$ *and* $(0 <)\theta^{(m)} \leqslant \theta_1(x), \theta_2(x)$ *on* $[x_1, x_2]$. *Then*

$$M \leqslant \int_{x_1}^{x_2} \frac{dx}{\theta(x)} + (3\theta^{(m)3})^{-1/2}((V^{(2/3)}(\theta_1; x_1, x_2))^{3/2} + (V^{(2/3)}(\theta_2; x_1, x_2))^{3/2}).$$

As in [4] we divide $[x_1, x_2]$ into n equal consecutive closed subintervals $\Delta_j = [x^{(j)}, x^{(j+1)}]$, $j = 1, \ldots, n$, and for $l = 1, 2$, let

$$\theta_l^{(s)}(x) = \min_{t \in \Delta_j} \theta_l(t) = \theta_l^{(j)}, \qquad x \in (x^{(j)}, x^{(j+1)}),$$

$$\theta_l^{(s)}(x^{(j)}) = \min(\theta_l^{(j-1)}, \theta_l^{(j)}), \qquad j = 2, \ldots, n,$$

$$\theta_l^{(s)}(x_1) = \min(\theta_l(x_1), \theta_l^{(1)}), \qquad \theta_l^{(s)}(x_2) = \min(\theta_l(x_2), \theta_l^{(n)}).$$

Let λ be a positive number. If $\theta_2^{(s)}(x^{(j)}) < \theta_2^{(j)}$, $j = 1, \ldots, n$, we take the graph

$$y - \theta_2^{(s)}(x^{(j)}) = \lambda(x - x^{(j)})^2, \qquad x \geqslant x^{(j)};$$

if $\theta_2^{(s)}(x^{(j)}) < \theta_2^{(j-1)}$, $j = 2, \ldots, n+1$, we take the graph

$$y - \theta_2^{(s)}(x^{(j)}) = \lambda(x - x^{(j)})^2, \qquad x \leqslant x^{(j)}.$$

If $\theta_1^{(s)}(x^{(j)}) < \theta_1^{(j)}$, $j = 1, \ldots, n$, we take the graph

$$y + \theta_1^{(s)}(x^{(j)}) = -\lambda(x - x^{(j)})^2, \qquad x \geqslant x^{(j)};$$

if $\theta_1^{(s)}(x^{(j)}) < \theta_1^{(j-1)}$, $j = 2, \ldots, n+1$, we take the graph

$$y + \theta_1^{(s)}(x^{(j)}) = -\lambda(x - x^{(j)})^2, \qquad x \leqslant x^{(j)}.$$

Let $\theta_2^{(t)}(x)$, $x_1 \leqslant x \leqslant x_2$, be the lower envelope of $\theta_2^{(s)}(x)$ and the first set of graphs; let $-\theta_1^{(t)}(x)$, $x_1 \leqslant x \leqslant x_2$, be the upper envelope of $-\theta_1^{(s)}(x)$ and the second set of graphs. There are decompositions of $[x_1, x_2]$ into sets of

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consecutive closed intervals $\Lambda_k^{(l)}$, $k = 1, \ldots, N_l$, $l = 1, 2$, on each of which $\theta_l^{(t)}(x)$ is given by an arc of one of the graphs or a segment of $\theta_l^{(s)}(x)$, each interval being maximal with this property. Let $\Lambda_k^{(l)} = [X_l^{(k)}, X_l^{(k+1)}]$.

The domain determined by $-\theta_1^{(t)}(x) < y < \theta_2^{(t)}(x)$, $x_1 < x < x_2$, becomes a quadrangle Q^* on assigning as a pair of opposite sides the segments

$$\sigma^*(x_l)\colon x = x_l, \qquad -\theta_1^{(t)}(x_l) < y < \theta_2^{(t)}(x_l), \qquad l = 1, 2.$$

For the module M^* of Q^* for the family of curves joining the pair of sides complementary to $\sigma^*(x_1)$, $\sigma^*(x_2)$ we have evidently $M \leqslant M^*$. We recall that in [4] we obtained for M^* the estimate (where $\theta^{(t)}(x) = \theta_1^{(t)}(x) + \theta_2^{(t)}(x)$)

$$(1)\quad M^* \leqslant \int_{x_1}^{x_2} \frac{dx}{\theta^{(t)}(x)} + \frac{1}{3}\int_{x_1}^{x_2} \frac{(\theta_1'^{(t)}(x))^2 - \theta_1'^{(t)}(x)\theta_2'^{(t)}(x) + (\theta_2'^{(t)}(x))^2}{\theta^{(t)}(x)} dx$$

which we can readily reduce to

$$(2)\qquad M^* \leqslant \int_{x_1}^{x_2} \frac{dx}{\theta^{(t)}(x)} + \frac{1}{2}\int_{x_1}^{x_2} \frac{(\theta_1'^{(t)}(x))^2 + (\theta_2'^{(t)}(x))^2}{\theta^{(t)}(x)} dx.$$

It should be observed that a result with some similarity to (1) was given earlier by Warschawski [5, Theorem IV(a)]. However it is not actually equivalent.

Writing $\theta^{(s)}(x) = \theta_1^{(s)}(x) + \theta_2^{(s)}(x)$ we have on the one hand,

$$\begin{aligned}
\int_{x_1}^{x_2} \frac{dx}{\theta^{(t)}(x)} - \int_{x_1}^{x_2} \frac{dx}{\theta^{(s)}(x)} &= \int_{x_1}^{x_2} \frac{\theta^{(s)}(x) - \theta^{(t)}(x)}{\theta^{(s)}(x)\theta^{(t)}(x)} dx \\
&\leqslant \frac{1}{(2\theta^{(m)})^2}\Bigg(\sum_{k=1}^{N_1} \int_{\Lambda_k^{(1)}} \left[\max_{x',x'' \in \Lambda_k^{(1)}} |\theta_1^{(s)}(x'') - \theta_1^{(s)}(x')|\right] dx \\
&\qquad + \sum_{k=1}^{N_2} \int_{\Lambda_k^{(2)}} \left[\max_{x',x'' \in \Lambda_k^{(2)}} |\theta_2^{(s)}(x'') - \theta_2^{(s)}(x')|\right] dx\Bigg).
\end{aligned}$$

The last term has the bound

$$\frac{\lambda^{-1/2}}{4\theta^{(m)2}}((V^{(2/3)}(\theta_1^{(s)}; x_1, x_2))^{3/2} + (V^{(2/3)}(\theta_2^{(s)}; x_1, x_2))^{3/2})$$

which in turn leads to the inequality

$$(3)\qquad \begin{aligned}
&\int_{x_1}^{x_2} \frac{dx}{\theta^{(t)}(x)} - \int_{x_1}^{x_2} \frac{dx}{\theta^{(s)}(x)} \\
&\qquad \leqslant \frac{\lambda^{-1/2}}{4\theta^{(m)2}}((V^{(2/3)}(\theta_1; x_1, x_2))^{3/2} + (V^{(2/3)}(\theta_2; x_1, x_2))^{3/2}).
\end{aligned}$$

On the other hand,

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$$\begin{aligned}\frac{1}{2}\int_{x_1}^{x_2}&\frac{(\theta_1'^{(t)}(x))^2+(\theta_2'^{(t)}(x))^2}{\theta^{(t)}(x)}\,dx\\&\leqslant\frac{1}{4\theta^{(m)}}\left(\sum_{k=1}^{N_1}\int_{\Lambda_k^{(1)}}(\theta_1'^{(t)}(x))^2\,dx+\sum_{k=1}^{N_2}\int_{\Lambda_k^{(2)}}(\theta_2'^{(t)}(x))^2\,dx\right)\\&\leqslant\frac{\lambda^{1/2}}{3\theta^{(m)}}((V^{(2/3)}(\theta_1;x_1,x_2))^{3/2}+(V^{(2/3)}(\theta_2;x_1,x_2))^{3/2}).\end{aligned}$$

Thus

$$M^*\leqslant\int_{x_1}^{x_2}\frac{dx}{\theta^{(t)}(x)}+\frac{\lambda^{1/2}}{3\theta^{(m)}}((V^{(2/3)}(\theta_1;x_1,x_2))^{3/2}+(V^{(2/3)}(\theta_2;x_1,x_2))^{3/2}).$$

Combining this with (3) and choosing λ to minimize the bound for M^* ($\lambda = 3/4\theta^{(m)}$), we obtain

$$\begin{aligned}M^*\leqslant\int_{x_1}^{x_2}&\frac{dx}{\theta^{(s)}(x)}\\&+(3\theta^{(m)3})^{-1/2}((V^{(2/3)}(\theta_1;x_1,x_2))^{3/2}+(V^{(2/3)}(\theta_2;x_1,x_2))^{3/2}).\end{aligned}$$

Letting n tend to infinity, we obtain the result of Theorem 1.

3. As in [4] we can derive at once from Theorem 1 a version of the Second Fundamental Inequality.

THEOREM 2. *Let D be a simply-connected domain in the z-plane with boundary elements P_1, P_2 such that for every x the segment $\sigma(x)$: $-\theta_1(x) < y < \theta_2(x)$, $\theta_1(x)$, $\theta_2(x) > 0$ separates D into subdomains with P_1, P_2 as respective boundary elements. Let $\theta(x) = \theta_1(x) + \theta_2(x)$. Let θ_1, θ_2 have finite 2/3-variation on any closed interval. Let $\theta_j(x) \leqslant L$, all $x, j = 1, 2$. Let*

$$\min_{x'\leqslant x\leqslant x''}(\theta_1(x),\theta_2(x)) = \theta^{(m)}(x',x'').$$

Let D be mapped conformally on the strip S: $0 < \mathfrak{I}\zeta < a$ in the ζ-plane so that P_1, P_2 correspond to the boundary elements of the latter determined by the point at infinity with respective neighborhoods in $\mathfrak{R}\zeta < 0$, $\mathfrak{R}\zeta > 0$. Let $\tau(x)$ denote the image of $\sigma(x)$ in S. Let

$$\xi_1(x) = \operatorname*{g.l.b.}_{\zeta\in\tau(x)}\mathfrak{R}\zeta,\qquad \xi_2(x) = \operatorname*{l.u.b.}_{\zeta\in\tau(x)}\mathfrak{R}\zeta.$$

Then for $x_1 < x_2$,

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$$\frac{1}{a}(\xi_2(x_2) - \xi_1(x_1)) \leqslant \int_{x_1}^{x_2} \frac{dx}{\theta(x)}$$

$$+ (3\theta^{(m)}(x_1, x_2))^{-1/2}((V^{(2/3)}(\theta_1; x_1, x_2))^{3/2} + (V^{(2/3)}(\theta_2; x_1, x_2))^{3/2})$$

$$+ \frac{2L}{\theta^{(m)}(x_1 - 2L, x_1 + 2L)} + \frac{2L}{\theta^{(m)}(x_2 - 2L, x_2 + 2L)}.$$

The proof is essentially that of [4, Theorem 3].

Bibliography

1. Lars Ahlfors, *Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen*, Acta Soc. Sci. Fenn. Nova Ser. **1** (1930), 1–40.

2. J. Ferrand and J. Dufresnoy, *Extension d'une inégalité de M. Ahlfors et application au problème de la dérivée angulaire*, Bull. Sci. Math. (2) **69** (1945), 165–174. MR **8**, 145.

3. F. Gehring, *A note on a paper by L. C. Young*, Pacific J. Math. **5** (1955), 67–72. MR **16**, 910.

4. James A. Jenkins and Kôtaro Oikawa, *On results of Ahlfors and Hayman*, Illinois J. Math. **15** (1971), 664–671. MR **45** #5332.

5. S. Warschawski, *On conformal mapping of infinite strips*, Trans. Amer. Math. Soc. **51** (1942), 280–335. MR **4**, 9.

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