PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 62, Number 2, February 1977

SPIN MANIFOLDS ARE DECOMPOSABLE

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ABSTRACT. It is shown that in the unoriented cobordism ring every manifold with Spin structure is decomposable.

1. **Introduction.** From the work of Thom [3], the unoriented cobordism ring $\mathfrak{N}_*$ is a polynomial ring over Z_2 on generators x_i of dimension i, with $i \neq 2^s - 1$, and the manifold M^i is indecomposable if and only if the characteristic number $S_i[M^i]$ is nonzero.

Dold [2] exhibited odd dimensional manifolds which are suitable generators, and these generators are in fact orientable manifolds. No even dimensional oriented manifold can be indecomposable, since

$$S_{2n}[M^{2n}] = \mathrm{Sq}^1 S_{2n-1}[M^{2n}] = w_1 S_{2n-1}[M^{2n}].$$

An examination of the known examples for generators quickly reveals that none admit Spin structures. This is not a coincidence, for one has

PROPOSITION. *Every Spin manifold is decomposable.*

2. **Proof.** Since a Spin manifold is oriented, even dimensional Spin manifolds are decomposable. Further, if M is a Spin manifold of dimension $4n + 1$,

$$S_{4n+1}[M] = \mathrm{Sq}^2 S_{4n-1}[M] = v_2 S_{4n-1}[M] = 0.$$

Now, let M be a Spin manifold of dimension $4n + 3$.

Claim. If $j \leqslant n$, and $x \in H^{n-j}(M; Z_2)$, then $x^4 S_{4j+3}[M] = 0$. This is clear for $j = 0$, since S_3 is zero, and inductively it may be assumed if $j' < j$.

Now $\mathrm{Sq}^1 S_{4j+3} = S_{4j+4} = S_{2j+2}^2 = \mathrm{Sq}^1(S_{2j+2}S_{2j+1})$, and by Proposition 6.1 of [1], ker Sq^1 = im Sq^1 in $H^*(B\,\mathrm{Spin}; Z_2)$ except in dimensions divisible by four. Thus $S_{4j+3} = S_{2j+2}S_{2j+1} + \mathrm{Sq}^1\sigma$ for some σ, and

$$x^4 S_{4j+3}[M] = x^4 S_{2j+2}S_{2j+1}[M] + \mathrm{Sq}^1(x^4\sigma)[M] = x^4 S_{2j+2}S_{2j+1}[M].$$

If j is odd, $2j + 1 = 4j' + 3$, $j' \leqslant j$ and

$$x^4 S_{2j+2}S_{2j+1}[M] = (xS_{(j+1)/2})^4 S_{4j'+3}[M] = 0.$$

If j is even,

Received by the editors July 6, 1976.
AMS (MOS) subject classifications (1970). Primary 57D75.
Key words and phrases. Cobordism, Spin manifold.
[1]The author is indebted to the National Science Foundation for financial support during this work.

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\begin{aligned}
x^4 S_{2j+2} S_{2j+1}[M] &= x^4 S_{2j+2} \mathrm{Sq}^2 S_{2j-1}[M] \\
&= x^4 \mathrm{Sq}^2(S_{2j+2}) S_{2j-1}[M] \\
&= x^4 S_{2j+4} S_{2j-1}[M] \\
&= (x S_{(j+1)/2})^4 S_{2j-1}[M],
\end{aligned}
$$

which is zero, since $2j - 1 = 4j' + 3$, $j' < j$.

References

1. D.W. Anderson, E.H. Brown, Jr. and F.P. Peterson, *The structure of the Spin cobordism ring*, Ann. of Math. (2) **86** (1967), 271–298. MR **36**#2160.

2. A. Dold, *Erzeugende der Thomschen Algebra* $\mathfrak{N}$, Math. Z. **65** (1956), 25–35. MR **18**, 60.

3. R. Thom, *Quelque propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86. MR **15**, 890.

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